

Self-reproduction in k -inflation

Ferdinand Helmer and Sergei Winitzki

*Arnold Sommerfeld Center, Department of Physics,
Ludwig-Maximilians University, Theresienstr. 37, 80333 Munich, Germany*

We study cosmological self-reproduction in models of inflation driven by a scalar field ϕ with a noncanonical kinetic term (k -inflation). We develop a general criterion for the existence of attractors and establish conditions selecting a class of k -inflation models that admit a unique attractor solution. We then consider quantum fluctuations on the attractor background. We show that the correlation length of the fluctuations is of order $c_s H^{-1}$, where c_s is the speed of sound. By computing the magnitude of field fluctuations, we determine the coefficients of Fokker-Planck equations describing the probability distribution of the spatially averaged field ϕ . The field fluctuations are generally large in the inflationary attractor regime; hence, eternal self-reproduction is a generic feature of k -inflation. This is established more formally by demonstrating the existence of stationary solutions of the relevant FP equations. We also show that there exists a (model-dependent) range $\phi_R < \phi < \phi_{\max}$ within which large fluctuations are likely to drive the field towards the upper boundary $\phi = \phi_{\max}$, where the semiclassical consideration breaks down. An exit from inflation into reheating without reaching ϕ_{\max} will occur almost surely (with probability 1) only if the initial value of ϕ is below ϕ_R . In this way, strong self-reproduction effects constrain models of k -inflation.

I. INTRODUCTION

The paradigm of k -inflation [1] assumes that the energy density in the early universe is dominated by a scalar field ϕ with an effective Lagrangian $p(X, \phi)$ consisting solely of a (noncanonical) kinetic term,

$$p(X, \phi) = K(\phi)L(X), \quad X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \quad (1)$$

where $K(\phi)$ and $L(X)$ are functions determined by the underlying fundamental theory. The effective Lagrangian (1) is assumed to be valid in a sufficiently wide range of values of ϕ and $\partial_\mu\phi$, as long as the energy density of the field ϕ is below a suitable scale (e.g. the Planck scale). Unlike the “traditional” models of inflation, where the dominant energy density is due to a potential $V(\phi)$, an accelerated expansion in k -inflation is driven by the kinetic energy of the scalar field.¹ The form (1) of the Lagrangian is sufficiently general to cover many interesting cases. For instance, Lagrangians of the form

$$p(X, \phi) = K_1(\phi)X^{n_1} + K_2(\phi)X^{n_2}, \quad (2)$$

where n_1, n_2 are fixed numbers, are reduced to the Lagrangian (1) after a suitable redefinition of the field ϕ .

The assumed initial conditions for k -inflation consist of the field ϕ having small spatial gradients within a sufficiently large initial region, and values of ϕ and the time derivative $\dot{\phi}$ within suitable (and wide) ranges. These

initial conditions guarantee that the energy density due to the field ϕ is positive and dominates other forms of energy in the universe. The evolving homogeneous field $\phi(t)$ rapidly approaches an attractor solution, $\phi_*(t)$, which drives inflation. The presence of the attractor makes a model largely insensitive to initial conditions, since the attractor solution is unique and is approached exponentially quickly. In the present paper we develop a general definition of attractor solutions and demonstrate their existence in a wide range of models of k -inflation.

The back-reaction of quantum fluctuations modifies this classical picture of cosmological evolution, adding random “jumps” to the classical “drift” along the attractor line. In generic models of potential-driven inflation, quantum fluctuations give rise to the phenomenon of eternal self-reproduction [6, 7, 8, 9, 10]. As soon as suitable initial conditions occur in one domain, ultimately an infinite volume of space will undergo a long period of inflation followed by reheating. In this way, the eternal character of the self-reproduction process significantly alleviates the problem of initial conditions for inflation.

The main focus of this paper is to investigate self-reproduction in generic models of k -inflation with Lagrangians of the form (1). We find that quantum “jumps” generically dominate over the “drift,” and that self-reproduction is generically present. Self-reproduction in inflation can be studied using the stochastic (“diffusion”) approach based on a Fokker-Planck (FP) equation (see Ref. [10] for a review). Although the FP approach depends on the choice of the spacelike foliation of space-time (“gauge”), one can use the FP equations to compute a number of gauge-independent characteristics, such as the fractal dimension of the inflating domain [11, 12] and the probability of reaching the end-of-inflation point $\phi = \phi_E$ of the configuration space. In particular, the presence of eternal self-reproduction is a gauge-invariant statement [12, 13]. We show that the fractal dimension of the inflating domain is generically close to 3, confirm-

¹ An effective scalar field ϕ with noncanonical kinetic terms is also used in models of k -essence ([2, 3], and more recently e.g. [4, 5], where the Lagrangian $p(X, \phi)$ is arranged to track the equation of state of other dominant matter components and to become dominant only at late times. In the present paper, we do not consider such models of k -essence but assume that the energy density of the field ϕ is dominant during inflation.

ing a strongly self-reproducing behavior in a wide class of models of k -inflation.

We also investigate the “stationarity” of the distribution of ϕ , in the sense of Refs. [10, 14, 15]. There exists a (model-dependent) boundary value ϕ_{\max} such that the stochastic formalism breaks down for $\phi > \phi_{\max}$. Imposing a boundary condition at $\phi = \phi_{\max}$, we find that the stationary volume-weighted distribution of the field ϕ is concentrated near ϕ_{\max} . In the terminology of Ref. [14], this constitutes a “runaway diffusion” behavior, meaning that the volume-averaged value of the field is driven to the imposed upper limit. Finally, we follow the evolution of the field ϕ along a single comoving worldline, starting at a generic value $\phi_0 \lesssim \phi_{\max}$. We find that the field will either reach the boundary $\phi = \phi_{\max}$ or, with roughly equal probability, exit the inflationary regime through the end-of-inflation (reheating) point $\phi = \phi_E$. An exit into reheating becomes overwhelmingly probable only if $\phi_0 < \phi_R$, where $\phi_R \ll \phi_{\max}$ is a model-dependent threshold value. Therefore, the semiclassical description of the inflationary evolution along a single comoving worldline will remain valid only if the initial value ϕ_0 is chosen within the interval $\phi_E < \phi_0 < \phi_R$. It is interesting to note that a similar conclusion was reached in a study of stochastic back-reaction effects in models of quintessence [16], where the initial value of the inflaton and the quintessence was found to be constrained to certain intervals. We compare this situation with the behavior of models of potential-driven inflation, where such a threshold $\phi_R \ll \phi_{\max}$ is absent and the exit into reheating has probability 1 for all $\phi_0 \lesssim \phi_{\max}$, where ϕ_{\max} is the Planck boundary.

Thus, the present investigation confirms that generic models of k -inflation exhibit standard features of eternal inflation, and shows that strong effects of self-reproduction constrain the choice of initial conditions.

II. MAIN RESULTS

In this section we present the main line of arguments and the principal results of this paper. Full details of the relevant derivations are delegated to sections III to V.

In the cosmological context, we consider a spatially homogeneous field $\phi = \phi(t)$ and a flat FRW metric,

$$g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)d\mathbf{x}^2, \quad (3)$$

where $a(t)$ is the scale factor. We also assume that the energy density of the field ϕ ,

$$\varepsilon(X, \phi) \equiv 2Xp_{,X} - p, \quad (4)$$

dominates all the other forms of energy in the universe. (Partial derivatives are denoted by a comma, so $p_{,X} \equiv \partial p / \partial X$.) Then, as is well known, the Einstein equations yield

$$\frac{\dot{a}}{a} \equiv H = \kappa\sqrt{\varepsilon}, \quad \kappa^2 \equiv \frac{8\pi G}{3} = \frac{8\pi}{3M_{Pl}^2}, \quad (5)$$

while the equation of motion for the field ϕ ,

$$\frac{d}{dt}(p_{,v}) + 3Hp_{,v} = p_{,\phi}, \quad v \equiv \dot{\phi} = \sqrt{2X}, \quad (6)$$

can be rewritten as [1]

$$\dot{\varepsilon} = -3H(\varepsilon + p). \quad (7)$$

In Sections III A–III D we determine the conditions for the existence of attractor solutions in general models of k -inflation. In a typical scenario, the field starts at a large initial value $\phi = \phi_0$ with $\dot{\phi} < 0$ and gradually approaches a smaller value $\phi = \phi_E$ where inflation ends. If an attractor exists, the trajectory in the phase space quickly approaches the attractor curve; the equation of motion for $\phi(t)$ is effectively reduced to a first-order equation. A slow-roll parameter can be introduced to quantify the deviation of the spacetime from exact de Sitter. In Sec. III E we show that a cosmological model with a Lagrangian of the form (1) admits an inflationary attractor solution under the following (sufficient) conditions:

(i) The function $K(\phi)$ remains positive at $\phi \rightarrow \infty$ and is such that $\int^\infty \sqrt{K(\phi)} d\phi = \infty$ (the integral diverges) and $\lim_{\phi \rightarrow \infty} (K^{-3/2} K') = 0$. For instance, functions $K(\phi) \sim \phi^s$ with $s > -2$ satisfy these two conditions.²

(ii) The equation $L'(X) = 0$ has a unique root $X_0 > 0$ such that the energy density, $\varepsilon(X_0, \phi) = -K(\phi)L(X_0)$, is positive and bounded away from zero for large ϕ . (This condition excludes $K(\phi) \sim \phi^s$ with $s < 0$.) The attractor solution is $X = X_*(\phi)$, where the function $X_*(\phi)$ tends to X_0 for large ϕ .

(iii) The speed of sound of field perturbations is real-valued, $c_s^2 > 0$, for $X = X_0$ and large ϕ (i.e. perturbations on the attractor background do not cause instabilities). It will be shown that $c_s^2 > 0$ if $L''(X_0)K'(\phi) > 0$ for large ϕ . The speed of sound c_s (which is small in typical models of k -inflation) plays the role of a slow-roll parameter.

The attractor solution will be $\dot{\phi} = v_*(\phi)$; as long as the system is on the attractor, all the cosmological functions can be expressed as functions of ϕ . We obtain a complete asymptotic expansion of $v_*(\phi)$ at large ϕ (Sec. III B) and a representation in terms of an integral equation (Sec. III C). We show that the slow-roll condition generally holds (Sec. III F). Explicit asymptotics in the leading slow-roll approximation are given in Sec. III G.

In the stochastic approach to self-reproduction, one is interested in averaged values of the field ϕ on sufficiently large distance scales, on which the fluctuations freeze and become uncorrelated. We show in Sec. IV B that the

² Technically, the two conditions are independent, as exemplified by the functions $K(\phi) = \phi^{-1}$ and $K(\phi) = (\phi \ln \phi)^{-2}$. Note that the borderline case $K(\phi) = \phi^{-2}$ must be considered separately (the attractor has the exact form $\dot{\phi} = \text{const}$). Below we shall only consider $K(\phi) \sim \phi^s$ with $s \geq 0$.

relevant scale L is not the Hubble horizon H^{-1} but the *sound horizon*, $L \sim c_s H^{-1}$, where the speed of sound is

$$c_s(X, \phi) = \sqrt{\frac{P_{,X}}{\varepsilon_{,X}}}. \quad (8)$$

Let us denote the averaging on these scales by an overbar. The dynamics of the averaged field $\bar{\phi}$ is a superposition of the deterministic motion and quantum “jumps” due to the backreaction of quantum fluctuations exiting the (sound) horizon. Since the attractor solution is approached exponentially quickly by other solutions, any deviations from the attractor are quickly suppressed. So we may assume that at every place the cosmological quantities such as H , c_s , are always given by the functions of ϕ derived from the attractor solution (Sec. III G).

The backreaction of quantum fluctuations on the averaged field $\bar{\phi}$ may lead to a qualitative change in the evolution of the field, compared with the deterministic trajectory $\dot{\phi} = v_*(\phi)$. To describe the dynamics of the averaged field during inflation, we use the stochastic or “diffusion” approach (see Ref. [10] for a review). The distribution $P_c(\phi, t)$ of values of $\bar{\phi}$ per *coordinate* (or comoving) volume satisfies the Fokker-Planck (FP) equation,

$$\frac{\partial P_c}{\partial t} = \frac{\partial^2}{\partial \phi^2} (D P_c) - \frac{\partial}{\partial \phi} (v_* P_c) \equiv \hat{L}_c P, \quad (9)$$

where $v_*(\phi)$ is the deterministic attractor trajectory and $D(\phi)$ is the “diffusion coefficient” describing the magnitude of fluctuations. The computation of $D(\phi)$ is performed in Sec. IV C, and the result (in the leading slow-roll approximation) is expressed as

$$D(\phi) \approx \text{const} \frac{H(\phi)}{c_s^3(\phi)}. \quad (10)$$

The comoving distribution P_c can be interpreted as the probability density, $P_c(\phi, t) d\phi$, for the value of ϕ at time t along a randomly chosen comoving worldline $\mathbf{x} = \text{const}$. Also of interest is the distribution of $\bar{\phi}$ weighted by *proper* 3-volume within 3-surfaces of equal time t . This volume-weighted distribution, denoted $P_p(\phi, t)$, satisfies an analogous FP equation, which differs from Eq. (9) only by the presence of the expansion term $3H(\phi)P$,

$$\frac{\partial P_p}{\partial t} = \frac{\partial^2}{\partial \phi^2} (D P_p) - \frac{\partial}{\partial \phi} (v_* P_p) + 3H P_p \equiv \hat{L}_p P. \quad (11)$$

These FP equations are supplemented by boundary conditions. One imposes an “exit-only” condition at the end-of-inflation point, $\phi = \phi_E$, and either a reflecting or an absorbing boundary condition at a suitable value $\phi = \phi_{\text{max}}$ (see Sec. V C for explicit details). One then expects that solutions of Eq. (9) should decay with time, indicating that at late times a given comoving worldline will have almost surely exited the inflationary regime. In Sec. V C, we show that the differential operator \hat{L}_c in the r.h.s. of Eq. (9) indeed has only negative eigenvalues, indicating a late-time behavior $P_c(\phi, t) \propto e^{-\lambda t}$ with

$\lambda > 0$. On the other hand, solutions of Eq. (11) may either decay or grow at late times, depending on whether the largest eigenvalue λ_{max} of the operator \hat{L}_p is negative or positive. A positive value of λ_{max} indicates the presence of eternal self-reproduction (eternal inflation), with a stationary distribution $P_p(\phi, t) \propto e^{\lambda_{\text{max}} t}$. It is found in Sec. V B that $\lambda_{\text{max}} > 0$ in generic models of k -inflation, and that the distribution $P_p(\phi, t)$ is concentrated near the upper boundary $\phi = \phi_{\text{max}}$. Moreover, we show that $\lambda_{\text{max}} \approx 3H_{\text{max}}$, where $H_{\text{max}} = H(\phi_{\text{max}})$ is the largest value of $H(\phi)$ within the allowed range of ϕ . This shows that the inflating domain grows almost at the rate of the Hubble expansion. It is known that the inflating domain can be visualized as a self-similar fractal set [11, 12]. The fact that $\lambda_{\text{max}} \approx 3H_{\text{max}}$ means that the fractal dimension of the inflating domain is close to 3, indicating a regime of strong self-reproduction.

Finally, we calculate the probability of reaching the end-of-inflation boundary ϕ_E if the evolution starts at an initial value $\phi = \phi_0$ (Sec. V D). In all models of k -inflation, the consistency of the diffusion approach requires one to limit the allowed range of ϕ by an upper boundary ϕ_{max} , even if the energy density does not reach the Planck scale (Sec. IV E). The boundary ϕ_{max} is analogous to the Planck boundary in models of potential-driven chaotic inflation, in that the semiclassical (“diffusion”) approach fails for $\phi > \phi_{\text{max}}$ because fluctuations become too large. One expects that the probability of exiting through the end-of-inflation point ϕ_E , rather than through the upper boundary ϕ_{max} , should be almost equal to 1, indicating that almost all comoving observers will exit inflation normally. We show that there exists a threshold value $\phi_R \ll \phi_{\text{max}}$ such that the exit through ϕ_E is overwhelmingly likely for initial values $\phi_0 < \phi_R$. (Such a threshold is absent in potential-driven inflation.) Hence, a given comoving worldline with $\phi_0 < \phi_R$ will almost surely (with probability 1) eventually exit inflation and enter the reheating phase. Nevertheless, the total proper 3-volume of the inflating domains will grow exponentially with time. Thus, we demonstrate that generic models of k -inflation exhibit the standard features of eternal self-reproduction.

III. ATTRACTOR SOLUTIONS IN k -INFLATION

In this section we study the cosmological dynamics of the spatially homogeneous field $\phi(t)$ with a Lagrangian $p(\dot{\phi}, \phi)$ and investigate the existence of attractor solutions in the regime $\phi \rightarrow \infty$. We show that attractor solutions exist in a broad class of models and determine the asymptotic forms of the attractors. As a particular example, we select the Lagrangian

$$p(\dot{\phi}, \phi) = K(\phi) Q(\dot{\phi}). \quad (12)$$

Since the equation of motion (6) does not depend explicitly on time t , it is convenient to consider solutions

$\phi(t)$ as curves in the phase plane (ϕ, v) , where we denote $v \equiv \dot{\phi}$. The equation of motion in the phase plane has the form

$$\frac{dv}{d\phi} = g(v, \phi), \quad (13)$$

where $g(v, \phi)$ is an auxiliary function expressed through the Lagrangian $p(v, \phi)$ as

$$g(v, \phi) = -\frac{1}{vp_{,vv}} \left(3\kappa \sqrt{\varepsilon(v, \phi)} p_{,v} + vp_{,v\phi} - p_{,\phi} \right), \quad (14)$$

$$\varepsilon(v, \phi) \equiv vp_{,v} - p. \quad (15)$$

In the particular case of the Lagrangian (12), the function $g(v, \phi)$ is given by

$$g(v, \phi) = -3\kappa \frac{Q'(v) \sqrt{\tilde{\varepsilon}(v)}}{vQ''(v)} \sqrt{K(\phi)} - \frac{\tilde{\varepsilon}(v)}{vQ''(v)} \frac{K'(\phi)}{K(\phi)}, \quad (16)$$

$$\tilde{\varepsilon}(v) \equiv \frac{\varepsilon(v, \phi)}{K(\phi)} = vQ'(v) - Q. \quad (17)$$

Here and below we assume that $K(\phi) > 0$ when $\phi \rightarrow \infty$. We can always change the sign of $Q(\dot{\phi})$ if this is not the case. Note that $K(\phi)$ may have a root at a finite ϕ , e.g. at the endpoint of inflation, but we only consider the inflationary regime where ϕ is sufficiently large and $K(\phi)$ remains positive.

A. Definition of attractors

Let us now motivate the definition of an attractor by analyzing the typical behavior of solutions in terms of the function $v(\phi)$. A sample set of trajectories in the phase plane (ϕ, v) is shown in Fig. 1. Trajectories starting at large ϕ and $v < 0$ will quickly approach an almost horizontal line, $v \approx v_0 = \text{const}$, and then proceed along that line more slowly towards $\phi = 0$. It is intuitively clear from the figure that the line $v \approx v_0$ (rather than any of the neighbor trajectories) should be considered the attractor solution. To make this statement precise, we need a formal criterion that would distinguish the attractor line from nearby solutions.

We begin by presenting a heuristic motivation for this criterion (which will be Eq. (18) below). In Fig. 1, the attractor solution stays approximately constant as $\phi \rightarrow -\infty$, while a generic neighbor trajectory $v_0 + \delta v(\phi)$ quickly moves away from the attractor. Let us therefore examine the growth of $\delta v(\phi)$ with ϕ as $\phi \rightarrow -\infty$. If $v \approx v_0$ is a solution of Eq. (13), then $g(v_0, \phi) \approx 0$ and $g(v_0 + \delta v, \phi) \approx g_{,v}(v_0, \phi) \delta v$. Since $\frac{d}{d\phi} \delta v \approx g_{,v} \delta v$, an initially small deviation $\delta v(\phi)$ grows exponentially with ϕ if $g_{,v}(v_0, \phi) \neq 0$. Assuming a power-law dependence $p(v, \phi) \sim v^n$ for large v , one finds from Eq. (14) that the function $g(v, \phi)$ will grow at least linearly with v at fixed ϕ ; the growth of $g(v, \phi)$ with v will be even faster if

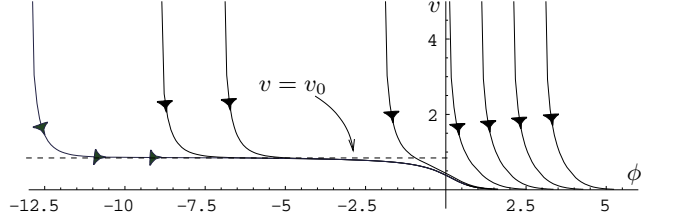


Figure 1: A numerically obtained phase plot for the Lagrangian $p(X, \phi) = (-X + X^2) \arctan \phi$. Trajectories starting at large negative ϕ with large positive $v \equiv \dot{\phi}$ approach an attractor solution. The attractor trajectory approximately coincides with the line $v = v_0$ for sufficiently large $|\phi|$.

$p(v, \phi)$ is exponential in v . Hence, the growth of a non-attractor solution for large ϕ is at least exponential, both for small and for large deviations from the attractor.

For convenience, below we always consider the quadrant $\phi > 0$ and $v < 0$ in the phase plane, since the opposite choice $\phi \rightarrow -\infty$ and $v > 0$ is treated completely analogously. In the general case, we expect that the attractor solution $v_*(\phi)$ will have a relatively slow behavior at $\phi \rightarrow \infty$, compared with an exponential behavior of nearby non-attractor solutions. Thus, a suitable condition for selecting the attractor solution is that $v_*(\phi)$ should grow slower than exponentially as $\phi \rightarrow \infty$, while nearby solutions grow exponentially. A formal way to express this condition is

$$\lim_{\phi \rightarrow \infty} \frac{d}{d\phi} \ln v_*(\phi) = 0. \quad (18)$$

We shall use Eqs. (13) and (18) as the *definition* of the attractor solution $v_*(\phi)$, together with the assumption that nearby trajectories do *not* satisfy Eq. (18). We stress that the attractor solution is singled out by an asymptotic condition at $\phi \rightarrow \infty$. At a finite value of ϕ , all the trajectories approach each other and no single solution appears to be special.

Let us remark that the definition (18) may be unduly restrictive. In particular, it does not allow attractors that are approached more slowly than exponentially. In such cases, the definition (18) may still be used to select the attractor solution after a suitable change of variable $\phi \rightarrow \tilde{\phi}$. Moreover, it may happen that either every solution or no solution of Eq. (13) satisfies Eq. (18). In such cases, a more detailed analysis is required to investigate the existence of attractors. Nevertheless, the condition (18) is adequate for the analysis of k -inflation.

B. Asymptotics in the general case

The next task is to determine the possible asymptotic behavior of the attractor $v_*(\phi)$, given the equation of motion (13). Let us assume that the function $g(v, \phi)$ has

an asymptotic expansion at $\phi \rightarrow \infty$ of the form

$$g(v, \phi) = A_0(v)B_0(\phi) + A_1(v)B_1(\phi) + \dots, \quad (19)$$

where $B_1(\phi)$ is subdominant to $B_0(\phi)$ as $\phi \rightarrow \infty$. The asymptotic expansion of $g(v, \phi)$ at $\phi \rightarrow \infty$ will have the form (19) if the Lagrangian $p(v, \phi)$ is polynomial in v . In particular, Eq. (16) is manifestly in this form.

In most cases, the term $A_0(v)B_0(\phi)$ in Eq. (19) will dominate at large ϕ , so that an approximate *general* (i.e. non-attractor) solution $v(\phi)$ of Eq. (13) is found as

$$\int_{v(\phi)}^{v_1} \frac{dv'}{A_0(v')} = \int_{\phi}^{\phi_1} B_0(\phi') d\phi', \quad (20)$$

where $v(\phi_1) = v_1$ is an arbitrary initial condition. The technical assumptions for the dominance of the term $A_0(v)B_0(\phi)$ are that both $A_0(v)$ and $A_1(v)$ are bounded for finite v and that $A_1(v)$ does not grow faster than $A_0(v)$ for large v . It is straightforward to verify, using Eq. (14), that these assumptions hold for a Lagrangian $p(\phi, v)$ which is polynomial in v .

To study the behavior of the general solution (20) for large ϕ , we need to consider two mutually exclusive cases: either $\int^{\infty} B_0(\phi) d\phi$ converges at $\phi \rightarrow \infty$, or it diverges. If this integral converges, there exist solutions $v(\phi)$ that approach any given value v_1 as $\phi \rightarrow \infty$, namely,

$$\int_{v(\phi)}^{v_1} \frac{dv'}{A_0(v')} = \int_{\phi}^{\infty} B_0(\phi') d\phi'. \quad (21)$$

Since v_1 can be chosen continuously, there are infinitely many solutions $v(t)$ that tend to a constant limit at $\phi \rightarrow \infty$. It follows that there is no unique attractor solution in this case. In the second case, $\int^{\infty} B_0(\phi) d\phi = \infty$, there may exist solutions $v(\phi)$ approaching a root of $A_0(v)$ as $\phi \rightarrow \infty$. To simplify the analysis, let us change the variables,

$$\phi \rightarrow \tilde{\phi}(\phi) \equiv \int_{\phi}^{\phi} B_0(\phi) d\phi, \quad (22)$$

noting that $\tilde{\phi} \rightarrow \infty$ together with $\phi \rightarrow \infty$. We can now rewrite the equation of motion through the new variable $\tilde{\phi}$ and look for attractor solutions at $\tilde{\phi} \rightarrow \infty$ using the same criteria as before. The asymptotic form of $\tilde{g}(v, \tilde{\phi}) \equiv g(v, \phi)/B_0(\phi)$ will be

$$\tilde{g}(v, \tilde{\phi}) = A_0(v) + A_1(v) \left. \frac{B_1(\phi)}{B_0(\phi)} \right|_{\phi \rightarrow \tilde{\phi}} + \dots \quad (23)$$

Essentially, the replacement $\phi \rightarrow \tilde{\phi}$ is equivalent to the assumption that $B_0(\phi) \equiv 1$ in Eq. (19). This is what we shall assume in this and the following sections.

If $v_*(\phi)$ is an attractor, Eq. (18) with $B_0(\phi) \equiv 1$ gives

$$0 = \lim_{\phi \rightarrow \infty} \frac{g(v_*(\phi), \phi)}{v_*(\phi)} = \lim_{\phi \rightarrow \infty} \frac{A_0(v_*(\phi))}{v_*(\phi)}, \quad (24)$$

and hence $v_*(\phi)$ approaches a root of $A_0(v)/v$ as $\phi \rightarrow \infty$. Thus, roots of the function $A_0(v)/v$ correspond to solutions that are approximately constant for large ϕ . Our previous analysis shows that such solutions, $v(\phi) \approx v_0$, will be attractors if $A'_0(v_0) > 0$ and repulsors if $A'_0(v_0) < 0$. Therefore, we focus attention on the case when the function $A_0(v)$ has a finite root $v_0 < 0$ such that $A'_0(v_0) > 0$, and we look for an attractor solution $v_*(\phi)$ that approaches $v = v_0$ when $\phi \rightarrow \infty$.³ Additionally, a physically meaningful inflationary attractor solution $v_*(\phi)$ must have a positive energy density, $\varepsilon(v_*(\phi), \phi) > 0$. Thus the root v_0 must also satisfy the condition

$$\lim_{\phi \rightarrow \infty} \varepsilon(v_0, \phi) > 0. \quad (25)$$

Since $v_*(\phi) \approx \text{const}$ for large ϕ , an approximation to the attractor solution can be found heuristically by neglecting v' in Eq. (13). Clearly, this is equivalent to the frequently used *slow-roll approximation* where one neglects $\ddot{\phi}$ and keeps only $\dot{\phi}$ in the equations of motion for $\phi(t)$. Thus, the slow-roll approximation $v_{sr}(\phi)$ to the attractor solution is the root of $g(v_{sr}(\phi), \phi) = 0$ which is near v_0 as $\phi \rightarrow \infty$.

We can now obtain a complete asymptotic expansion of the attractor solution $v_*(\phi)$ at $\phi \rightarrow \infty$. Since $\lim_{\phi \rightarrow \infty} g_{,v}(v, \phi) = A'_0(v) \neq 0$ in a neighborhood of v_0 , we may solve Eq. (13) with respect to v at least in some neighborhood of $(\phi = \infty, v = v_0)$ and obtain

$$v(\phi) = g^{-1}\left(\frac{dv}{d\phi}, \phi\right), \quad (26)$$

where $g^{-1}(\cdot, \phi)$ is the inverse of $g(v, \phi)$ with respect to v . (If the equation $g(v, \phi) = 0$ has several roots with respect to v , an appropriate choice of the branch of the inverse function must be made to ensure that $g^{-1}(0, \phi) \rightarrow v_0$ as $\phi \rightarrow \infty$). Let us apply the method of iteration to Eq. (26), starting with the initial approximation $v_{(0)}(\phi) = \text{const}$. The first approximation will coincide with the slow-roll solution,

$$v_{(1)}(\phi) = g^{-1}(0, \phi) \equiv v_{sr}(\phi). \quad (27)$$

³ Thus, we omit the consideration of cases where the existence of attractors is less straightforward to establish. For instance, the function $A_0(v)/v$ may have no finite roots, may approach zero as $v \rightarrow \infty$, may have a root $v_0 = 0$, or a higher-order (multiple) root. Also, an attractor solution may exist not due to a root of $A_0(v)$ but due to a cancellation among several terms in the expansion (19). Finally, an attractor solution may grow without bound, $v_*(\phi) \rightarrow \infty$ as $\phi \rightarrow \infty$, instead of approaching a constant. In every such case, a more detailed analysis is needed to investigate the existence of attractors, possibly involving a further change of variable, such as $v(\phi) \equiv \tilde{v}(\phi)F(\phi)$, where $F(\phi)$ is a function with a suitable asymptotic behavior at $\phi \rightarrow \infty$. See, for instance, Appendix A where we use this technique to establish the existence of unique attractors in models of potential-driven inflation.

We shall now show that the successive approximations $v_{(n)}(\phi)$, $n = 1, 2, \dots$, differ from the exact attractor by terms of progressively higher asymptotic order at $\phi \rightarrow \infty$. It will follow that the sequence $\{v_n(\phi)\}$ converges to the attractor in the *asymptotic* sense (i.e. for $\phi \rightarrow \infty$ at fixed n). This asymptotic approximation is similar in spirit to the asymptotic expansion using a hierarchy of slow-roll parameters, which was developed in Ref. [17].

By construction, successive approximations $v_{(n)}(\phi)$ satisfy

$$v_{(n+1)}(\phi) = g^{-1} \left(\frac{dv_{(n)}(\phi)}{d\phi}, \phi \right). \quad (28)$$

We denote by $v_*(\phi)$ the *exact* attractor solution satisfying Eq. (26). Then the deviation of $v_{(n)}$ from v_* satisfies the recurrence relation

$$\begin{aligned} v_{(n+1)} - v_* &= g^{-1}(v'_{(n)}, \phi) - g^{-1}(v'_*, \phi) \\ &= \frac{1}{g_{,v}(v_*, \phi)} [v_{(n)} - v_*]_{,\phi} + O[(v_{(n)} - v_*)^2]. \end{aligned} \quad (29)$$

Since $v_{(1)}(\phi) - v_*(\phi)$ decays at $\phi \rightarrow \infty$ (this is so because both functions approach the same constant v_0), it follows that $(v_{(n)} - v_*)$ also *decays* at $\phi \rightarrow \infty$ for all n . Hence, the terms $(v_{(n)} - v_*)_{,\phi}$ and $(v_{(n)} - v_*)^2$ have a higher asymptotic order (i.e. decay faster as $\phi \rightarrow \infty$) than $(v_{(n)} - v_*)$. Since for large ϕ we have $g_{,v}(v_*) \approx A'_0(v_0)$, which is a nonzero constant, it follows that $(v_{(n+1)} - v_*)$ has a higher asymptotic order than $(v_{(n)} - v_*)$ at $\phi \rightarrow \infty$.

C. Existence of attractors

In the previous section, we have shown that an asymptotic expansion of the attractor solution may be obtained by iterating Eq. (26). However, the successive terms of the expansion will contain derivatives of $B_1(\phi)$ of successively higher orders. Since the n -th derivative of an analytic function usually grows as quickly as $\sim n!$, the asymptotic expansion could be a divergent series. Such a series cannot be used to reconstruct the attractor solution exactly and establish its existence. In fact, the asymptotic property of the sequence $v_{(n)}(\phi)$ was derived *assuming* that the exact attractor solution $v_*(\phi)$ exists.

The existence of the attractor solution can be established by producing a convergent sequence based on an integral equation. To this end, we rewrite the function $g(v, \phi)$ as

$$g(v, \phi) = (v - v_0) \alpha + (v - v_0)^2 A(v) + B(v, \phi), \quad (30)$$

where $\alpha \equiv A'_0(v_0) > 0$ and $A(v), B(v, \phi)$ are auxiliary functions such that $A(v)$ is regular at $v = v_0$ and $B(v, \phi)$ decays as $\phi \rightarrow \infty$ uniformly for all v near v_0 . (It is possible to express $g(v, \phi)$ in the above form because $g_{,v}(v_0, \phi) \rightarrow \alpha$ as $\phi \rightarrow \infty$. The function $B(v, \phi)$ has the stated property since $A_1(v)$ is regular at v_0 .) A trivial redefinition $\phi \rightarrow \alpha^{-1}\phi$ effectively sets $\alpha = 1$, which

simplifies the analysis. Then Eq. (13) with the boundary condition $v(\infty) = v_0$ is equivalent to the following integral equation,

$$\begin{aligned} v(\phi) &= v_0 - e^\phi \int_\phi^\infty e^{-\phi'} [(v(\phi') - v_0)^2 A(v(\phi')) \\ &\quad + B(v(\phi'), \phi')] d\phi'. \end{aligned} \quad (31)$$

This equation can be iterated starting from the initial approximation $v^{(0)}(\phi) \equiv v_0$, which produces successive approximations $v^{(n)}(\phi)$, $n = 1, 2, \dots$. We shall now show that the sequence $v^{(n)}(\phi)$ converges as $n \rightarrow \infty$ for sufficiently large (but fixed) ϕ . The limit of the sequence will be, evidently, the attractor solution $v_*(\phi)$.

We shall prove the convergence of the sequence $\{v^{(n)}(\phi)\}$ by showing that the difference between successive approximations, $|v^{(n+1)}(\phi) - v^{(n)}(\phi)|$, decays with n at fixed ϕ . Let us first estimate the difference $\Delta_n(\phi) \equiv v^{(n)}(\phi) - v_0$ at some fixed ϕ . By construction, we have

$$\begin{aligned} \Delta_{n+1}(\phi) &= -e^\phi \int_\phi^\infty e^{-\phi'} [\Delta_n^2 A(v^{(n)}(\phi')) + B(v^{(n)}(\phi'), \phi')] d\phi'. \end{aligned} \quad (32)$$

To determine a bound for Δ_n using this equation, we need some bounds on the functions $A(v)$ and $B(v, \phi)$ valid near $v = v_0$ and for large ϕ . By assumption, $B(v, \phi)$ decays uniformly in ϕ , so for any small number $\delta > 0$ we can choose ϕ large enough, say $\phi > \phi_0$, so that

$$\left. \begin{aligned} |B(v, \phi)| &< \delta \\ |B_{,v}(v, \phi)| &< \delta \end{aligned} \right\} \quad \text{for } |v - v_0| < \delta \text{ and } \phi > \phi_0. \quad (33)$$

Since $A(v)$ is regular at $v = v_0$, there exists a constant $C > 0$ such that $|A(v)| < C$ for $|v - v_0| < \delta$. Then the difference Δ_{n+1} is estimated using Eq. (32) as follows,

$$|\Delta_{n+1}(\phi)| \leq C |\Delta_n(\phi)|^2 + \delta. \quad (34)$$

Since the initial approximation is $v^{(0)}(\phi) \equiv v_0$, it is easy to see that, with $\delta < (4C)^{-1}$, we have

$$|\Delta_n(\phi)| < \frac{1 - \sqrt{1 - 4C\delta}}{2C} < 2\delta, \text{ for } \forall n \text{ and } \phi > \phi_0. \quad (35)$$

Let us choose $\delta < (4C)^{-1}$ and find a corresponding value of ϕ_0 such that Eq. (33) holds. Then we compute

$$\begin{aligned} &|v^{(n+1)}(\phi) - v^{(n)}(\phi)| \\ &\leq \int_\phi^\infty e^{\phi - \phi'} \left[|(v^{(n)} - v_0)^2 - (v^{(n-1)} - v_0)^2| C \right. \\ &\quad \left. + |v^{(n)} - v^{(n-1)}| \delta \right] d\phi'. \end{aligned} \quad (36)$$

Since

$$\begin{aligned} &|(v^{(n)} - v_0)^2 - (v^{(n-1)} - v_0)^2| \\ &= |(v^{(n)} - v^{(n-1)})(v^{(n)} - v_0 + v^{(n-1)} - v_0)| \\ &< 4 |v^{(n)} - v^{(n-1)}| \delta, \end{aligned} \quad (37)$$

we have

$$\begin{aligned} & \left| v^{(n+1)}(\phi) - v^{(n)}(\phi) \right| \\ & \leq 5\delta \int_{\phi}^{\infty} e^{\phi-\phi'} \left| v^{(n)}(\phi') - v^{(n-1)}(\phi') \right| d\phi'. \end{aligned} \quad (38)$$

Now we use induction in n , starting with

$$\begin{aligned} \left| v^{(1)}(\phi) - v^{(0)}(\phi) \right| & \leq \int_{\phi}^{\infty} d\phi' e^{\phi-\phi'} B(v_0, \phi) \\ & < \int_{\phi}^{\infty} d\phi' e^{\phi-\phi'} \delta = \delta, \end{aligned} \quad (39)$$

to establish the bound

$$\left| v^{(n+1)}(\phi) - v^{(n)}(\phi) \right| < (5\delta)^n \delta, \text{ for } \forall n \text{ and } \phi > \phi_0. \quad (40)$$

This bound tends to zero for $n \rightarrow \infty$ as long as $\delta < \frac{1}{5}$. If necessary, we can diminish δ so that $\delta < \frac{1}{5}$ and choose a corresponding value of ϕ_0 such that Eq. (33) holds. Then the sequence $v^{(n)}(\phi)$ converges as $n \rightarrow \infty$ at any fixed $\phi > \phi_0$.

D. Summary of assumptions

In the preceding sections we have established that a general model of k -inflation admits an attractor solution $v_*(\phi)$ under the following (sufficient) conditions:

1) The function $g(v, \phi)$ defined by Eq. (14) has an asymptotic expansion at $\phi \rightarrow \infty$ of the form (19), where $B_1(\phi)/B_0(\phi) \rightarrow 0$ as $\phi \rightarrow \infty$. This asymptotic expansion defines the functions $A_0(v)$, $A_1(v)$.

2) The function $A_0(v)$ has a unique root $v_0 < 0$ such that $A'_0(v_0) > 0$.

3) The function $A_1(v)$ is regular at v_0 and does not grow faster than $A_0(v)$ for large v .

4) The function $B_0(\phi)$ is such that $\int^{\infty} B_0(\phi) d\phi = \infty$ (the integral diverges).

We have shown that the attractor solution $v_*(\phi)$ approaches v_0 as $\phi \rightarrow \infty$ and has a well-defined asymptotic expansion for $\phi \rightarrow \infty$, which can be determined by iterating Eq. (26) starting with $v(\phi) \equiv v_0$. The first iteration yields the slow-roll approximation to the attractor, which is the function $v_{sr}(\phi)$ defined by

$$g(v_{sr}(\phi), \phi) = 0. \quad (41)$$

Assuming that (generically) $A_1(v_0) \neq 0$, we can obtain the following leading-order *approximate* expression for the slow-roll solution,

$$v_{sr}(\phi) \approx v_0 - \frac{A_1(v_0)}{A'_0(v_0)} \frac{B_1(\phi)}{B_0(\phi)}. \quad (42)$$

In addition to the above assumptions, the following two conditions must be satisfied in order for the attractor solution to be physically relevant: (i) The energy density,

$\varepsilon(v, \phi) \equiv v p_{,v} - p$, is positive for all v near v_0 and does not approach zero as $\phi \rightarrow \infty$. (ii) The speed of sound is real-valued, $c_s^2 > 0$. The condition (i) will be satisfied if $\varepsilon(v_0, \phi) > 0$ for all sufficiently large ϕ . The condition (ii) says that

$$c_s^2 = \frac{p_{,v}}{\varepsilon_{,v}} = \frac{p_{,v}}{v p_{,vv}} \Big|_{v=v_0} > 0. \quad (43)$$

An explicit form of the condition (43) for the Lagrangian (12) is given in Eq. (52) below.

E. Existence of attractors in k -inflation

We now apply the preceding constructions to a model with the Lagrangian (12). We first need to analyze the asymptotic form of the equation of motion, $v' = g(v, \phi)$, at $\phi \rightarrow \infty$. The function $g(v, \phi)$ is given by Eq. (16), which is in the form (19) with only two terms,

$$g(v, \phi) = A_0(v) B_0(\phi) + A_1(v) B_1(\phi), \quad (44)$$

$$A_0(v) \equiv -3\kappa \frac{Q'(v) \sqrt{\tilde{\varepsilon}(v)}}{v Q''(v)}, \quad A_1(v) \equiv -\frac{\tilde{\varepsilon}(v)}{v Q''(v)}, \quad (45)$$

$$\tilde{\varepsilon}(v) \equiv v Q'(v) - Q(v); \quad (46)$$

$$B_0(\phi) \equiv \sqrt{K(\phi)}, \quad B_1(\phi) \equiv \frac{d}{d\phi} [\ln K(\phi)]. \quad (47)$$

The asymptotic behavior of the functions $B_0(\phi)$ and $B_1(\phi)$ at $\phi \rightarrow \infty$ may belong to one of the three cases: (i) $B_0(\phi) \ll B_1(\phi)$, (ii) $B_0(\phi) \gg B_1(\phi)$, and (iii) $B_0(\phi) \sim B_1(\phi)$. In the first case, the equation of motion is dominated by $A_1(v)$ which cannot have roots $A_1(v_0) = 0$, otherwise the energy density would become small on the attractor, $\varepsilon(v_*, \phi) \approx \varepsilon(v_0, \phi) = 0$ at $\phi \rightarrow \infty$, and could not dominate the energy density of other matter. Hence, in case (i) inflationary attractors do not exist. Case (ii) covers all functions $K(\phi)$ that either grow with ϕ , or tend to a constant, or decay slower than ϕ^{-2} as $\phi \rightarrow \infty$. Case (iii) corresponds to a specific family of Lagrangians with $K(\phi) \propto \phi^{-2}$, which gives $B_0(\phi) \sim B_1(\phi) \propto \phi^{-1}$. After a change of variable, $\phi \rightarrow \tilde{\phi} \equiv \ln \phi$, this case is reduced to case (ii) with $B_0 = 1$ and $B_1 = 0$. Therefore, we shall confine our attention to case (ii). The precise conditions for that case are

$$\frac{B_1(\phi)}{B_0(\phi)} \equiv \frac{K'(\phi)}{K^{3/2}(\phi)} \rightarrow 0 \text{ as } \phi \rightarrow \infty; \quad (48)$$

$$\int^{\infty} B_0(\phi) d\phi = \int^{\infty} \sqrt{K(\phi)} d\phi = \infty. \quad (49)$$

Assuming a polynomial growth of $Q(v) \sim v^n$ for large v , where $n > 2$, we find that $A_0(v)$ grows faster than $A_1(v)$; this verifies condition 3 in Sec. III D. Therefore, the term $A_0(v) B_0(\phi)$ dominates the equation of motion for non-attractor solutions at large ϕ and, as shown in previous sections, an attractor solution $v_*(\phi)$ must tend to a root $v_0 \neq 0$ of $A_0(v)$ as $\phi \rightarrow \infty$. The function $A_0(v)$

is expressed by Eq. (45). Since, as we already found, $\tilde{\varepsilon}(v)$ must be nonzero to drive inflation, it follows that v_0 must be a root of $Q'(v)$. Let us assume that $v_0 < 0$ is such a root, $Q'(v_0) = 0$. It remains to verify the other conditions for the existence of the attractor.

The energy density is positive on the attractor solution, $\varepsilon = -K(\phi)Q(v_0) > 0$, if $K(\phi) > 0$ and $Q(v_0) < 0$. Thus, the only admissible roots of $Q'(v_0)$ are such that $K(\phi)Q(v_0) < 0$. The condition $A'_0(v_0) > 0$ is satisfied since $v_0 < 0$, $\tilde{\varepsilon} > 0$, and thus

$$A'_0(v_0) = -3\kappa\sqrt{\tilde{\varepsilon}(v_0)}\frac{Q''(v_0)}{v_0Q''(v_0)} = -3\kappa\frac{\sqrt{\tilde{\varepsilon}}}{v_0} > 0, \quad (50)$$

where we have assumed that (generically) $Q''(v_0) \neq 0$. Finally, the speed of sound must satisfy the condition (43),

$$c_s^2 = \frac{p,v}{vp,vv} = \frac{Q'(v_*)}{v_*Q''(v_*)} \approx \frac{v_*(\phi) - v_0}{v_0} > 0. \quad (51)$$

This condition effectively constrains the deviation of the attractor solution $v_*(\phi)$ from $v = v_0 = \text{const}$, requiring that $v_*(t) < v_0$. Because of that, we have $Q'(v_*) < 0$ and $g(v_*, \phi) > 0$. Then the equation of motion, $dv_*/d\phi = g(v_*, \phi)$, entails that the negative value $v_*(\phi)$ monotonically increases (in the algebraic sense) as ϕ decreases, at least until the slow-roll approximation $v_*(\phi) \approx v_0$ breaks down. Numerical simulations for some model Lagrangians show that $v_*(\phi)$ is typically monotonic in ϕ all the way until the end of inflation.

Using the approximate solution (42) and the fact that $v_0 < 0$, we rewrite the condition (51) as

$$c_s^2 = \frac{1}{|v_0|} \frac{A_1(v_0)}{A'_0(v_0)} \frac{B_1(\phi)}{B_0(\phi)} = \frac{\sqrt{\tilde{\varepsilon}}}{3\kappa|v_0|Q''(v_0)} \frac{K'(\phi)}{K^{3/2}(\phi)} > 0. \quad (52)$$

It follows that $c_s^2 > 0$ if $Q''(v_0)K'(\phi) > 0$ for large ϕ . We conclude that a model of k -inflation satisfying the conditions listed in Sec. II will admit an attractor solution with properties suitable for inflationary cosmology.

F. Slow-roll condition

In this section we show that attractor solutions in k -inflation models satisfy the slow-roll condition for large ϕ . The slow-roll condition means that the change in the Hubble rate, $\Delta H \equiv \dot{H}\Delta t$, is negligible during one Hubble time $\Delta t = H^{-1}$, i.e.

$$H^{-1} \left| \dot{H} \right| \ll H. \quad (53)$$

This condition guarantees that the spacetime is locally sufficiently close to de Sitter during inflation.

It follows from Eq. (7) that

$$\left| \frac{1}{H} \frac{dH}{dt} \right| = \frac{1}{2} \left| \frac{1}{\varepsilon} \frac{d\varepsilon}{dt} \right| = \frac{3}{2} H \left| \frac{\varepsilon + p}{\varepsilon} \right|, \quad (54)$$

therefore the slow-roll condition with the Lagrangian (12) is equivalent to $|\varepsilon + p| \ll \varepsilon$ or

$$\left| \frac{vp,v}{vp,v - p} \right| = \left| \frac{vQ'(v)}{vQ'(v) - Q(v)} \right| \ll 1. \quad (55)$$

Since the attractor approaches a root v_0 of $Q'(v)$ as $\phi \rightarrow \infty$, and since $Q(v_0) < 0$ by the condition of the positivity of the energy density, the slow-roll condition will be satisfied for large enough ϕ . More precisely, ϕ must be large enough so that

$$\frac{v_0Q''(v_0)}{Q(v_0)} |v(\phi) - v_0| \ll 1. \quad (56)$$

The slow-roll condition is violated (and inflation ends) when $\ddot{a} = 0$, which is equivalent to $\dot{H} + H^2 = 0$ or $\varepsilon + 3p = 0$. This occurs at a time t when either $vQ'(v) + 2Q(v) = 0$ with $v \equiv \dot{\phi}(t)$, or $K(\phi(t)) = 0$ (then $\varepsilon = p = 0$ at the same time). Generically, we may expect that the former condition does not take place; so the end-of-inflation point is the root ϕ_E of $K(\phi_E) = 0$. For instance, this is the case when $K(\phi)$ is monotonic and $Q(v) = -c_1v^{n_1} + c_2v^{n_2}$ with $c_1 > 0$, $c_2 > 0$, $n_2 > n_1 > 1$.

G. Asymptotics of attractors in k -inflation

Assuming that the conditions for the existence of the attractor are met, let us now compute the asymptotic form of the attractor solution for large ϕ . All the functions describing the homogeneous cosmological solutions ($\dot{\phi}$, H , c_s , etc.) can be expressed as functions of ϕ . Substituting Eqs. (45)-(47) into Eq. (42), we find

$$\dot{\phi} \equiv v_*(\phi) \approx v_0 - v_1(\phi), \quad (57)$$

$$v_1(\phi) \equiv \frac{\sqrt{-Q(v_0)}}{3\kappa Q''(v_0)} \frac{K'(\phi)}{K^{3/2}(\phi)}, \quad (58)$$

$$\phi(t) \approx \phi_0 + v_0 t + \frac{2}{3\kappa Q''(v_0)} \sqrt{\frac{-Q(v_0)}{K(\phi_0 + v_0 t)}}, \quad (59)$$

$$H(\phi) \approx \kappa \sqrt{-K(\phi)Q(v_0)} \left(1 + \frac{v_0 Q''(v_0)}{2Q(v_0)} v_1(\phi) \right), \quad (60)$$

$$\ln a(\phi) = \int_{\phi_0}^{\phi} \frac{H}{v} d\phi \approx \frac{\kappa \sqrt{-Q(v_0)}}{|v_0|} \int_{\phi_0}^{\phi} \sqrt{K(\phi)} d\phi, \quad (61)$$

$$c_s^2(\phi) \approx -\frac{v_1(\phi)}{v_0} = \frac{\sqrt{-Q(v_0)}}{3\kappa|v_0|Q''(v_0)} \frac{K'(\phi)}{K^{3/2}(\phi)}. \quad (62)$$

The assumed conditions are

$$v_0 < 0, \quad Q'(v_0) = 0, \quad Q(v_0) < 0, \quad (63)$$

$$K(\phi) > 0, \quad K'(\phi)Q''(v_0) > 0. \quad (64)$$

Note that $c_s \ll 1$ for large ϕ , due to the condition (48). The small value of c_s^2 can be considered a “slow-roll parameter,” i.e. a parameter describing the smallness of the

deviation from the exact de Sitter evolution. The slow-roll condition (56) holds if

$$\frac{v_0^2 Q''(v_0)}{Q(v_0)} c_s^2(\phi) = \frac{|v_0|}{3\kappa\sqrt{|Q(v_0)|}} \frac{K'(\phi)}{K^{3/2}(\phi)} \ll 1. \quad (65)$$

This inequality determines a model-dependent slow-roll range $\phi > \phi_{sr}$. The slow-roll approximation becomes increasingly precise in the large ϕ limit where $c_s \rightarrow 0$.

IV. MAGNITUDE OF QUANTUM FLUCTUATIONS

In this section we compute the magnitude of quantum fluctuations of the field ϕ . This calculation will yield the relevant kinetic coefficients for the Fokker-Planck equation, which will be considered in Sec. V.

A. Quantization of fluctuations

We consider cosmological perturbations of the field ϕ on the inflationary attractor solution in a spatially flat FRW universe. We follow Ref. [18], where the quantum theory of perturbations for fields with noncanonical kinetic terms was developed.

Perturbations of the field ϕ give rise to scalar perturbations of the metric. Their simultaneous dynamics is described by a single scalar field u with the (classical) equation of motion

$$u'' - c_s^2 \Delta u - \frac{z''}{z} u = 0, \quad (66)$$

where c_s is the speed of sound, the prime denotes derivatives with respect to the conformal time,

$$u' \equiv \frac{\partial u}{\partial \eta} \equiv a(t) \frac{\partial u}{\partial t}, \quad (67)$$

and the auxiliary function $z(t)$ is defined by

$$z \equiv \frac{a\sqrt{\varepsilon + p}}{c_s H}. \quad (68)$$

The fluctuation $\delta\phi$ of the field ϕ and the Newtonian gravitational potential Φ are expressed through u (in the longitudinal gauge) as [18]

$$\frac{\delta\phi}{\dot{\phi}} = \frac{u}{zH} - \frac{\Phi}{H}, \quad (69)$$

$$\frac{\Phi}{H} = \frac{3\kappa^2}{2} z^2 \Delta^{-1} \frac{d}{dt} \left(\frac{u}{z} \right). \quad (70)$$

It is convenient to consider the Fourier modes $\delta\phi_k$, Φ_k , u_k of the perturbation variables; then the inverse Laplace operator in the above equation becomes simply $-k^{-2}$.

For a model with the Lagrangian (12) and the attractor solution found in Sec. III G, we have

$$z = \frac{1}{\kappa} |v_0| a(t) \sqrt{\frac{Q_0''}{|Q_0|}} (1 + O(c_s^2)), \quad (71)$$

$$zH = |v_0| a(t) \sqrt{K(\phi) Q_0''} (1 + O(c_s^2)), \quad (72)$$

where we denoted for brevity $Q_0 \equiv Q(v_0)$, $Q_0'' \equiv Q''(v_0)$. We find that $z(t) \approx a(t) \cdot \text{const}$; here and below we denote by “ \approx ” the approximation obtained by neglecting terms of order $c_s^2 \ll 1$. Within this approximation, we can express the field fluctuation $\delta\phi_k$ as

$$\begin{aligned} \delta\phi_k &= \frac{v_0}{zH} u_k - \frac{3\kappa^2}{2} v_0 z^2 \Delta^{-1} \frac{d}{dt} \left(\frac{u_k}{z} \right) \\ &\approx -\frac{u_k}{a\sqrt{KQ_0''}} - \frac{3\kappa a^2 v_0^2}{2k^2} \sqrt{\frac{Q_0''}{|Q_0|}} \frac{d}{dt} \left(\frac{u_k}{a} \right). \end{aligned} \quad (73)$$

It is known [18] that the action functional for the field $u(\mathbf{x}, \eta)$ has the canonical form. Therefore, one quantizes the field $u(\mathbf{x}, \eta)$ by postulating the mode expansion

$$\hat{u}(\mathbf{x}, \eta) = \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} [\hat{a}_{\mathbf{k}} u_{\mathbf{k}}^*(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} + H.c.], \quad (74)$$

where $\hat{a}_{\mathbf{k}}$ is a canonical annihilation operator satisfying $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta(\mathbf{k} - \mathbf{k}')$, and “ $H.c.$ ” denotes Hermitian conjugate terms. The mode functions $u_{\mathbf{k}}(\eta)$ are normalized by the condition $\text{Im}(\dot{u}_{\mathbf{k}} u_{\mathbf{k}}^*) = 1$ and satisfy

$$u_{\mathbf{k}}'' + c_s^2 k^2 u_{\mathbf{k}} - \frac{a''}{a} u_{\mathbf{k}} = 0, \quad (75)$$

where we replaced z''/z by a''/a , which is justified within the slow-roll range. The conformal time variable η can be expressed as a function of ϕ ,

$$\begin{aligned} \eta &= - \int_0^\phi \frac{d\phi}{|v_0| a(\phi)} \\ &\approx - \int_0^\phi \frac{d\phi}{|v_0|} \exp \left[\frac{\kappa \sqrt{|Q_0|}}{|v_0|} \int^\phi \sqrt{K(\phi')} d\phi' \right] \\ &\approx - \frac{1}{\kappa \sqrt{|Q_0|} K(\phi)} \exp \left[\frac{\kappa \sqrt{|Q_0|}}{|v_0|} \int^\phi \sqrt{K(\phi')} d\phi' \right]. \end{aligned} \quad (76)$$

It follows that $a(\eta) \approx (H\eta)^{-1}$ and $a''/a \approx -2\eta^{-2}$, so the mode functions $u_{\mathbf{k}}(\eta)$ can be chosen as the standard Bunch-Davies mode functions for the massless field (except for the extra factors c_s),

$$u_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{c_s k}} e^{i c_s k \eta} \left(1 + \frac{i}{c_s k \eta} \right). \quad (77)$$

Since the field modes $\delta\phi_k$ are linearly related to u_k , a similar mode expansion holds for the quantum field

$\delta\hat{\phi}(\mathbf{x}, t)$, except for different mode functions as given by Eq. (73):

$$\begin{aligned}\delta\hat{\phi}(\mathbf{x}, \eta) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} [\hat{a}_{\mathbf{k}}\delta\phi_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + H.c.], \quad (78) \\ \delta\phi_k(\eta) &= -\frac{u_k}{a\sqrt{K(\phi)Q_0''}} - \frac{3\kappa a^2 v_0^2}{2k^2} \sqrt{\frac{Q_0''}{|Q_0|}} \frac{d}{dt} \left(\frac{u_k}{a}\right) \\ &\approx \kappa\eta u_k(\eta) \left[\sqrt{\frac{|Q_0|}{Q_0''}} + \frac{3}{2} \frac{v_0^2 c_s^2}{1 - ic_s k \eta} \sqrt{\frac{Q_0''}{|Q_0|}} \right]. \quad (79)\end{aligned}$$

We may now disregard the term of order c_s^2 and obtain

$$\delta\phi_k(\eta) \approx \kappa\eta u_k(\eta) \sqrt{\frac{|Q_0|}{Q_0''}} = \kappa \frac{\eta e^{ic_s k \eta}}{\sqrt{c_s k}} \left[1 + \frac{i}{c_s k \eta} \right] \sqrt{\frac{|Q_0|}{Q_0''}}. \quad (80)$$

We conclude that in the slow-roll regime the mode function $\delta\phi_k(\eta)$ is proportional to the standard Bunch-Davies mode function evaluated at the wavenumber $c_s k$.

B. Scales of averaging

We are now interested in the spatially averaged perturbations $\delta\bar{\phi}$ of the field ϕ . It is well known [7] that quantum fluctuations of a scalar field (in the Bunch-Davies vacuum state) in de Sitter spacetime become uncorrelated on Hubble distance and time scales, $L \sim \delta t \sim H^{-1}$, and are transformed into classical perturbations. In the present case, the relevant distance scale is the sound horizon scale, $L \sim c_s H^{-1}$, while the time scale remains unchanged, $\delta t \sim H^{-1}$. Although this was already noted e.g. in Ref. [18], we shall now justify this statement more formally using arguments similar to those of Ref. [19].

The averaged fluctuation field $\delta\bar{\phi}$ can be defined as

$$\delta\bar{\phi}(\mathbf{x}, t) = \int d^3\mathbf{x}' \delta\hat{\phi}(\mathbf{x}', t) W(L^{-1}|\mathbf{x} - \mathbf{x}'|), \quad (81)$$

where L is the averaging scale and $W(q)$ is a suitable window function that quickly decays for $q \gtrsim 1$. Then the effective noise field is defined as

$$\bar{\xi} \equiv \frac{d}{dt} \delta\bar{\phi}. \quad (82)$$

For a minimally coupled, massless scalar field ϕ , it was shown in Ref. [19] that correlators of the noise field $\bar{\xi}$ have the following large-distance asymptotics, which hold for a wide class of window functions W :

$$\langle \bar{\xi}(\mathbf{x}, t) \bar{\xi}(\mathbf{x}', t') \rangle \sim \frac{e^{-2H|t-t'|}}{H^4 |\mathbf{x} - \mathbf{x}'|^4}. \quad (83)$$

This indicates a quick decay on time and distance scales of order the Hubble horizon. In the present calculation,

the field $\delta\hat{\phi}$ differs from the minimally coupled massless field considered in Ref. [19] only by the replacement $k \rightarrow c_s k$ and by an overall factor; this is easy to see by examining the mode function (80). Therefore, the calculations of Ref. [19] apply also to the present case if we rescale the length as $L \rightarrow c_s^{-1} L$. The time scale remains unchanged, whereas the distance scale is found from the relation $c_s^{-1} L = H^{-1}$, which yields $L = c_s H^{-1}$.

C. Diffusion coefficient

The effective dynamics of the averaged field $\bar{\phi}$ is described by a Langevin equation,

$$\frac{d\bar{\phi}}{dt} = v_*(\bar{\phi}) + \bar{\xi}(\mathbf{x}, t), \quad (84)$$

where $\bar{\xi}(\mathbf{x}, t)$ is the effective “noise” field, which encapsulates the effects of subhorizon quantum fluctuations. In the stochastic formalism, the “noise” is treated as a *classical* field, i.e. a Gaussian random function with known correlations (“colored noise”) [19, 20, 21]. It was shown in Sec. IV B that the field $\bar{\xi}$ is essentially uncorrelated beyond Hubble time scales $\delta t \sim H$ and beyond distance scales $L \sim c_s H^{-1}$. To simplify the analysis, one may treat the “noise” field $\bar{\xi}$ approximately as white noise as long as one considers variations of $\bar{\phi}$ only on sufficiently large distance and time scales. Such variations of $\bar{\phi}$ can then be described by the following Langevin equation, which is this time a *difference* equation,

$$\bar{\phi}(\mathbf{x}, t + \delta t) - \bar{\phi}(\mathbf{x}, t) = v_*(\bar{\phi})\delta t + \sqrt{2D(\bar{\phi})\delta t}\delta\gamma, \quad (85)$$

where D is a “diffusion coefficient” and $\delta\gamma$ is a fiducial Gaussian random variable with unit variance. The coefficient $D(\phi)$ is defined through the standard deviation of the quantum “jump” in the field $\bar{\phi}$ during a time interval δt ,

$$2D\delta t \equiv \langle \delta\bar{\phi}^2(\mathbf{x}, t) \rangle_{L \sim c_s H^{-1}} \equiv \left\langle \left(\frac{d}{dt} \delta\bar{\phi} \cdot \delta t \right)^2 \right\rangle_{L \sim c_s H^{-1}}, \quad (86)$$

where we explicitly indicate averaging the field $\frac{d}{dt} \delta\bar{\phi}(\mathbf{x}, t)$ on distance scales $L \sim c_s H^{-1}$.

As long as one considers variations of $\bar{\phi}$ on scales at least $\sim H^{-1}$, the difference equation (85) is equivalent to the stochastic differential equation

$$\frac{d\bar{\phi}}{dt} = v_*(\bar{\phi}) + \sqrt{2D(\bar{\phi})}\gamma, \quad (87)$$

where γ is a fiducial “white noise” field. The above equation is, in turn, equivalent to a FP equation for the time-dependent probability distribution $P_c(\phi, t)$,

$$\frac{\partial P_c}{\partial t} = \frac{\partial^2}{\partial \phi^2} (D P_c) - \frac{\partial}{\partial \phi} (v_* P_c) \equiv \hat{L}_c P, \quad (88)$$

where we use the Ito factor ordering in the FP equation [22].

We can now compute the diffusion coefficient $D(\phi)$ in the presently considered model of k -inflation, using Eq. (86). Field fluctuations on distance scales $L \sim c_s H^{-1}$ in the Bunch-Davies vacuum state are due to field modes exiting the sound horizon. Therefore, the magnitude of these fluctuations can be estimated by evaluating the absolute value of the mode function (80) at the wavenumber k that corresponds to the sound horizon crossing, $c_s k = aH \approx |\eta|^{-1}$. Since the quantum field $\frac{d}{dt}\delta\hat{\phi}$ has the standard mode expansion with mode functions $\frac{d}{dt}\delta\phi_k$, we find using Eq. (80) that

$$\frac{d}{dt}\delta\phi_k = a^{-1} \frac{d}{d\eta} \kappa \frac{\eta e^{ic_s k \eta}}{\sqrt{c_s k}} \left[1 + \frac{i}{c_s k \eta} \right] \sqrt{\frac{|Q_0|}{Q_0''}} \quad (89)$$

$$= \kappa \sqrt{\frac{|Q_0|}{Q_0''}} \frac{e^{ic_s k \eta}}{\sqrt{c_s k}} i c_s k \eta^2 H. \quad (90)$$

Therefore, the typical fluctuation in $\frac{d}{dt}\delta\hat{\phi}$ averaged on scales $L \sim c_s H^{-1}$ and accumulated during one Hubble time, $\delta t = H^{-1}$, is

$$\begin{aligned} \langle \delta\bar{\phi}^2 \delta t^2 \rangle_{L \sim c_s H^{-1}} &\approx \left(\frac{k^3}{4\pi^2} \left| \frac{d}{dt} \delta\phi_k \right|^2 H^{-2} \right) \Big|_{c_s k |\eta|=1} \\ &= \frac{\kappa^2}{4\pi^2} \frac{|Q_0|}{Q_0''} c_s^{-3}. \end{aligned} \quad (91)$$

Finally, the coefficient $D(\phi)$ is computed from Eq. (86),

$$D = \frac{1}{2\delta t} \left\langle \left(\frac{d}{dt} \delta\bar{\phi} \cdot \delta t \right)^2 \right\rangle_{L \sim c_s H^{-1}} \approx \frac{\kappa^2}{8\pi^2} \frac{|Q_0|}{Q_0''} \frac{H}{c_s^3}. \quad (92)$$

Since $c_s \ll 1$ for most of the allowed range of ϕ , the diffusion coefficient is typically large. As we shall see, this leads to the presence of self-reproduction.

A more explicit expression for $D(\phi)$ can be found using Eqs. (60) and (62),

$$D(\phi) = \frac{\kappa^{9/2} |Q_0|^{3/4} |v_0|^{3/2} \sqrt{27Q_0''} [K(\phi)]^{11/4}}{8\pi^2 [K'(\phi)]^{3/2}}. \quad (93)$$

D. Conditions for self-reproduction

Qualitatively, self-reproduction is present if the typical change in the averaged field due to random quantum jumps (“diffusion”) during one Hubble time is much larger than the deterministic change due to the classical equation of motion,

$$\sqrt{2D(\phi)\delta t} \gg v(\phi)\delta t, \quad \delta t \equiv H^{-1}. \quad (94)$$

It is straightforward to see that this condition generically holds in the slow-roll regime. Using Eq. (92), we find

$$\frac{(v\delta t)^2}{D\delta t} \sim \frac{v_0^2}{DH} \sim \frac{c_s^3}{H^2} \rightarrow 0 \quad \text{as } \phi \rightarrow \infty, \quad (95)$$

since $c_s(\phi) \rightarrow 0$, whereas $H(\phi) \sim \sqrt{K(\phi)}$ remains constant or grows as $\phi \rightarrow \infty$. Therefore, diffusion will dominate over the deterministic drift for all sufficiently large ϕ . Moreover, since the slow-roll condition is essentially the same as $c_s^2 \ll 1$, diffusion becomes negligible only near the end of the slow-roll range or even beyond that, near the end of inflation. A wide range of ϕ where diffusion dominates over the deterministic drift is a qualitative indication of eternal self-reproduction. A precise criterion for the presence of self-reproduction is provided by the Fokker-Planck equation for the volume-weighted distribution of the field, which will be analyzed in Sec. V.

E. Validity of diffusion approximation

We have computed the magnitude $\delta\phi$ of a typical random “jump”, defined as $\delta\phi \sim \sqrt{2D(\phi)\delta t}$, during one Hubble time $\delta t = H^{-1}$. The diffusion approximation is valid if $\delta\phi$ is “small” in the sense that it causes small changes in the energy density of the field,

$$\delta\varepsilon \equiv \varepsilon_{,\phi} \delta\phi \ll \varepsilon(\phi). \quad (96)$$

If this condition does not hold, we would not be justified in assuming that the “jumps” cause a perturbatively small change in the local expansion rate H . Since $\varepsilon_{,\phi}/\varepsilon \sim H_{,\phi}/H$, an equivalent condition is

$$\frac{H_{,\phi}}{H} \delta\phi \equiv \frac{H_{,\phi}}{H} \sqrt{2D(\phi)H^{-1}} \ll 1. \quad (97)$$

There is another aspect that limits the validity of the diffusion approach. As we have seen, the “jumps” are uncorrelated on distances of order $c_s H^{-1}$, which is significantly smaller than the Hubble scale H^{-1} since $c_s \ll 1$. A comoving scale $c_s H^{-1}$ grows to Hubble size within the time $\delta t_g = H^{-1} \ln c_s^{-1}$. Thus, a single Hubble-sized region will contain small inhomogeneities in the values of ϕ on scales above $c_s H^{-1}$, generated during the time interval $\delta t_g \gtrsim H^{-1}$. The typical magnitude of these inhomogeneities,

$$\delta\phi_g \sim \sqrt{2D(\phi)\delta t_g} = \sqrt{2DH^{-1} \ln c_s^{-1}}, \quad (98)$$

should not lead to a significant variation of H within a Hubble-sized region. (Otherwise, we cannot approximate the spacetime as a locally homogeneous de Sitter with a well-defined value of H .) Thus we obtain the condition

$$\frac{H_{,\phi}}{H} \delta\phi_g \equiv \frac{H_{,\phi}}{H} \sqrt{\ln c_s^{-1}} \delta\phi = \sqrt{2DH^{-1} \ln c_s^{-1}} \frac{H_{,\phi}}{H} \ll 1, \quad (99)$$

which is stronger than the condition (97) by the factor $\sqrt{\ln c_s^{-1}}$. However, this factor is an extremely slowly growing function of ϕ , which we may expect to contribute at most a factor of 2 or 3, and definitely no more than one order of magnitude. Therefore, we may ignore this factor and concentrate on the condition (97).

To investigate whether the condition (97) may be violated at large ϕ where $v_*(\phi) \approx v_0$, we can use the analytic approximations derived above. Using the formulas (60), (62), and (92), we find

$$\delta\phi \sim \sqrt{2DH^{-1}} = \frac{\kappa}{4\pi} \sqrt{\frac{-Q_0}{Q_0''}} c_s^{-3/2} = b_1 \frac{K^{9/8}(\phi)}{(K_{,\phi})^{3/4}}, \quad (100)$$

$$b_1 \equiv \frac{\kappa^{7/4}}{4\pi} (-Q_0)^{1/8} (Q_0'')^{1/4} [3|v_0|]^{3/4}. \quad (101)$$

The constant coefficient b_1 is small since it contains a large power of the inverse Planck mass, while other relevant energy scales are presumably much lower. Since $(\ln H)_{,\phi} = (\ln K)_{,\phi}$, the condition (97) yields

$$b_1 \left(K_{,\phi} \sqrt{K} \right)^{1/4} \ll 1. \quad (102)$$

Assuming the asymptotic behavior $K(\phi) \sim \phi^s$ for large ϕ , where $s \geq 0$, we simplify the above condition to

$$b_1 \phi^{\frac{3}{8}s - \frac{1}{4}} \ll 1. \quad (103)$$

This condition is satisfied for large ϕ if $0 \leq s \leq \frac{2}{3}$ but leads to an upper bound, $\phi < \phi_{\max}$, on values of ϕ if $s > \frac{2}{3}$. In the latter case, for $\phi > \phi_{\max}$ the fluctuations are so large that a single “quantum jump” (within a Hubble time $\delta t \sim H^{-1}$) may bring the value of ϕ outside of the slow-roll range, e.g. reach the end of inflation. It is clear that the diffusion approximation breaks down for $\phi > \phi_{\max}$. In other words, regions of the universe with $\phi > \phi_{\max}$ cannot be described within the present semiclassical framework because quantum fluctuations are too large. A boundary condition needs to be imposed at $\phi = \phi_{\max}$ if we wish to proceed using the diffusion approach. One may impose an absorbing boundary condition, arguing that regions with $\phi > \phi_{\max}$ disappear into a “sea of eternal randomness.” Alternatively, one may impose a reflecting boundary condition, arguing that the “sea of randomness” emits and absorbs equally many regions. Since this boundary condition serves only to validate the diffusion approach, results can be trusted only if they are insensitive to the chosen type of the boundary condition.

Models with $K(\phi) \sim \phi^s$ and $0 \leq s \leq \frac{2}{3}$ do not exhibit the problem described above, however it is still necessary to impose a boundary condition at some value $\phi = \phi_{\max}$. For instance, the energy density for large ϕ may reach the Planck value, which determines a (model-dependent) Planck boundary ϕ_{\max} such that $K(\phi_{\max})|Q_0| \sim M_{Pl}^4$. Results of applying the diffusion approach should be insensitive to the type of the boundary condition as well as to the precise value of ϕ_{\max} .

V. ANALYSIS OF FOKKER-PLANCK EQUATIONS

A. Self-adjoint form of the FP equation

As is long known, the FP equation (11) can be reduced to a manifestly self-adjoint form, which is formally similar to a Schrödinger equation for a particle in a one-dimensional potential. One can then use standard results about the existence of bound states, which correspond to stationary solutions of the FP equation.

A stationary solution of the FP equation is of the form

$$P_p(\phi, t) = P_{(\lambda)}(\phi) e^{\lambda t}, \quad (104)$$

where $P_{(\lambda)}(\phi)$ is a stationary probability distribution. The function $P_{(\lambda)}(\phi)$ must be everywhere positive, integrable, and satisfy the stationary FP equation,

$$\hat{L}_p P_{(\lambda)} \equiv (DP_{(\lambda)})_{,\phi\phi} - (v_* P_{(\lambda)})_{,\phi} + 3HP_{(\lambda)} = \lambda P_{(\lambda)}. \quad (105)$$

The proper time t may be replaced by another time variable, $d\tau = T(\phi)dt$, where $T(\phi)$ is an arbitrary positive function of ϕ , and it is implied that the value of τ is obtained by integrating $\int T(\phi)dt$ along comoving worldlines [23]. The FP equation in the time gauge τ has the same form as Eq. (105), except that the coefficients D, v, H are divided by $T(\phi)$. For instance, with $T(\phi) = H(\phi)$ we obtain the “scale factor” or “ e -folding” time variable $\tau = \int H dt = \ln a$, and the stationary FP equation for $P_{(\lambda)}^{(sf)}(\phi)$ is

$$\left(\frac{D}{H} P_{(\lambda)}^{(sf)} \right)_{,\phi\phi} - \left(\frac{v_*}{H} P_{(\lambda)}^{(sf)} \right)_{,\phi} + 3P_{(\lambda)}^{(sf)} = \lambda P_{(\lambda)}^{(sf)}. \quad (106)$$

Self-reproduction is eternal if there exists a stationary solution $P_{(\lambda)}$ of Eq. (105) with a positive λ . This condition is independent of the time gauge within the class of gauges $d\tau = T(\phi)dt$ [12].

To investigate the existence of positive eigenvalues of the operator \hat{L}_p , it is convenient to transform the stationary FP equation into an explicitly self-adjoint form. We perform the calculation following Ref. [23] and introduce new variables as follows,

$$x \equiv x(\phi) \equiv \int^\phi \frac{d\phi}{\sqrt{D(\phi)}}, \quad \partial_\phi = \frac{1}{\sqrt{D}} \partial_x, \quad (107)$$

$$P_{(\lambda)}(\phi) \equiv \psi(x) D^{-3/4}(\phi) \exp \left[\frac{1}{2} \int^\phi \frac{v_*(\phi)}{D(\phi)} d\phi \right], \quad (108)$$

where the replacement $\phi \rightarrow \phi(x)$ is implied in the last line. Note that $P_{(\lambda)}(\phi)$ is being multiplied by a strictly positive function of ϕ , so the positivity of $\psi(x)$ will entail the positivity of $P_{(\lambda)}(\phi)$. Under the replacements (107)-(108), Eq. (105) is transformed into

$$\psi_{,xx} - U(x)\psi = \lambda\psi, \quad (109)$$

where $U(x)$ is the “potential” defined as a function of ϕ through the relation

$$U(x) \equiv -3H(\phi) + \frac{3}{16} \frac{(D, \phi)^2}{D} - \frac{D, \phi \phi}{4} - \frac{v_* D, \phi}{2D} + \frac{v_*, \phi}{2} + \frac{v_*^2}{4D}. \quad (110)$$

Equation (109) is the self-adjoint form of the stationary FP equation and is formally equivalent to a stationary Schrödinger equation for a particle in a one-dimensional “potential” $U(x)$ with the “energy” $E \equiv -\lambda$. It is well-known that the ground state of a Schrödinger equation can be chosen as a nonnegative wavefunction. If we show that Eq. (109) has a ground state $\psi(x)$ with a negative value of “energy” $E_0 < 0$, it will follow that there exists a stationary probability distribution $P_{(\lambda_{\max})}(\phi)$ corresponding to the largest eigenvalue $\lambda_{\max} > 0$ of the FP equation. Then the existence of eternal self-reproduction will be established.

In order to find out whether the ground state energy in a given potential $U(x)$ is negative, it is convenient to use the variational principle,

$$E_0 = \min_{\psi(x)} \frac{\int \bar{\psi} (-\psi_{,xx} + U\psi) dx}{\int \bar{\psi} \psi dx}. \quad (111)$$

We may now substitute any test function $\psi(x)$ and obtain an upper bound on E_0 . If we find such $\psi(x)$ that the integral in the numerator of Eq. (111) is negative, it will follow that $E_0 < 0$. It is sufficient to consider real-valued $\psi(x)$ that vanish at the boundaries of the range of integration. Integrating by parts and omitting boundary terms, we find

$$\int \psi (-\psi_{,xx} + U\psi) dx = \int [(\psi_{,x})^2 + U\psi^2] dx. \quad (112)$$

If $U(x)$ is everywhere positive, then clearly $E_0 > 0$, so the only possibility to have a negative E_0 is to have a range of x where $U(x) < 0$. Let us suppose that $U(x) < 0$ within a range $x_1 < x < x_2$, with a typical value $U(x) \sim -U_0 < 0$ in the middle of the range. Then we may choose a test function $\psi(x)$ so that $\psi(x) = 0$ outside of the range $[x_1, x_2]$ and $\psi(x) \sim 1$ within that range. The typical value of $\psi'(x)$ will be of order $(x_2 - x_1)^{-1} \equiv l^{-1}$, and so we obtain a bound

$$E_0 \leq \frac{\int (\psi'^2 + U\psi^2) dx}{\int \psi^2 dx} \sim \frac{l^{-1} - U_0 l}{l} = -U_0 + \frac{1}{l^2}. \quad (113)$$

The desired negative bound will be achieved if the width l of the interval is sufficiently large so that $l^2 > U_0^{-1}$. However, the width l is constrained by

$$l \leq \frac{U_0}{\max_{x \in [x_1, x_2]} |U'(x)|}, \quad (114)$$

since the function $U(x)$ must vary between 0 and $-U_0$ within the interval of width l . Hence,

$$E_0 \leq -U_0 + \left[\frac{\max_{x \in [x_1, x_2]} |U'(x)|}{U_0} \right]^2. \quad (115)$$

In the next section, we shall use this condition near a point x where $U(x) = -U_0 < 0$.

B. Stationary solutions in k -inflation

Let us now analyze a model of k -inflation with the Lagrangian (12), where

$$K(\phi) \sim \phi^s, \quad s \geq 0, \quad (116)$$

for large ϕ . We assume that the conditions for the existence of an inflationary attractor are met. Let us now estimate the “potential” $U(x)$ at values of x that correspond to large ϕ . (Note that an infinite range of ϕ may be mapped into a finite range of x .) It is more convenient to analyze the behavior of the “potential” U as a function of ϕ . We find the following asymptotic behavior of parameters at $\phi \rightarrow \infty$:

$$H(\phi) \sim \phi^{s/2}, \quad v(\phi) \approx v_0 = \text{const}, \quad (117)$$

$$c_s(\phi) \sim \phi^{-\frac{1}{2} - \frac{s}{4}}, \quad (118)$$

$$D(\phi) \sim \phi^{n(s)}, \quad n(s) \equiv \frac{3}{2} + \frac{5}{4}s. \quad (119)$$

The dominant negative term in $U(x)$ is $-3H(\phi) \sim -\phi^{s/2}$, while the dominant positive term could be only

$$\frac{3}{16} \frac{(D, \phi)^2}{D} - \frac{D, \phi \phi}{4} \sim n(4-n)\phi^{n-2}. \quad (120)$$

However, the condition (97), rewritten as

$$\sqrt{DH^{-1}} \frac{H, \phi}{H} \ll 1, \quad (121)$$

together with the power-law dependences of $D(\phi)$, $H(\phi)$, yields

$$D, \phi \phi \sim \frac{D}{\phi^2}, \quad H, \phi \sim \frac{H}{\phi}, \quad (122)$$

$$\frac{D, \phi \phi}{H} \sim \frac{D}{H} \frac{1}{\phi^2} \sim DH^{-1} \left(\frac{H, \phi}{H} \right)^2 \ll 1. \quad (123)$$

Since the condition (121) must be satisfied for the entire range of ϕ under consideration, the term (120) is negligible compared with $-3H$. Similarly, the other terms decay as ϕ^{-1} or faster for large ϕ ; for instance, $v_*^2/D \ll H$ within the entire slow-roll regime (see Sec. IV D). Therefore, the potential $U(x)$ can be estimated as $U(x) \approx -3H(\phi)$ for most of the allowed range of ϕ , excluding only a narrow range near the end of inflation.

We now use the condition (115) to show that the ground state energy corresponding to the potential $U(x)$ is negative. Since $dx/d\phi > 0$, and since the function $H(\phi) \sim \phi^{s/2}$ grows monotonically, the largest value of $H_{,x}$ is at the largest allowed ϕ . The bound (115) applied to an interval $[\phi_1, \phi_2]$, upon using $dx = \sqrt{D}d\phi$, yields

$$\begin{aligned} E_0 &< -3H(\phi_2) + D(\phi_2) \left(\frac{H, \phi}{H} \right)^2 \\ &\approx -3H(\phi_2) + D(\phi_2) \phi_2^{-2} \approx -3H(\phi_2) < 0, \end{aligned} \quad (124)$$

since $D\phi^{-2} \ll H$ according to Eq. (123). Thus we have shown that the ground state has negative “energy.” This proves that the largest eigenvalue λ_{\max} of the operator \hat{L}_p is positive, and hence that eternal self-reproduction is present.

The bound $E_0 < -3H(\phi_2)$, where ϕ_2 is essentially any value within the allowed range $\phi_{sr} < \phi < \phi_{\max}$, means that the largest eigenvalue λ_{\max} is approximately equal to $3H_{\max}$, where H_{\max} is the largest accessible value of $H(\phi)$. It follows that the fractal dimension of the inflating domain (defined in a gauge-invariant way in Ref. [12]) is generically close to 3.

Finally, we note that the potential $U(x) \approx -3H(\phi)$ has a global minimum near the upper boundary ϕ_{\max} . It follows that the ground state $\psi(x)$ will have a sharp maximum near the upper boundary. Therefore, the stationary distribution $P_{(\lambda_{\max})}(\phi)$ indicates that most of the 3-volume contains $\phi \sim \phi_{\max}$. However, it has been long known that the distributions $P_c(\phi, t)$ and $P_p(\phi, t)$ are sensitive to the choice of equal-time hypersurfaces, or the “time gauge” (see e.g. [10, 14, 23, 24, 25, 26]). For instance, the 3-volume may be dominated by $\phi \sim \phi_{\max}$ in one gauge but not in another; even the exponential growth of the 3-volume of the inflating domain is gauge-dependent [13]. Nevertheless, the distributions $P_c(\phi, t)$ and $P_p(\phi, t)$ provide a useful qualitative picture of the global features of the spacetime during eternal inflation.

C. Eigenvalues of the comoving FP equation

We now consider the comoving distribution $P_c(\phi, t)$, which describes the values of ϕ along a single, randomly chosen comoving worldline. This distribution satisfies Eq. (9),

$$\frac{dP_c}{dt} = \hat{L}_c P_c, \quad (125)$$

which can be brought into a self-adjoint form similarly to the proper-volume FP equation. However, let us show directly that all the eigenvalues of the operator \hat{L}_c (with appropriate boundary conditions) are negative. It will follow that the late-time asymptotic of Eq. (125) is

$$P_c(\phi, t) = P_{(\lambda_0)}(\phi) e^{\lambda_0 t}, \quad \lambda_0 < 0. \quad (126)$$

In other words, the probability $P_c(\phi, t)$ tends uniformly to zero at late times, which is interpreted to mean that the evolution of ϕ along any particular comoving worldline will almost surely (with probability 1) eventually arrive at one of the boundaries, either at ϕ_E or at ϕ_{\max} .

Equation (125) can be written as a “conservation law,”

$$\partial_t P_c = \partial_\phi J, \quad J(\phi) \equiv \partial_\phi(DP_c) - v_* P_c, \quad (127)$$

where J plays the role of the “current.” The relevant types of boundary conditions, which were discussed above, are conveniently expressed in terms of the quantity J . Namely, the reflecting condition is $J = 0$, the

absorbing condition is $P_c = 0$, and the “exit-only” condition (to be imposed at the end-of-inflation point ϕ_E) is $J = -v_* P_c$, meaning that diffusion cannot bring the field from $\phi = \phi_E$ back into the inflationary range. To avoid considering the absorbing and the reflecting conditions separately, we shall impose at $\phi = \phi_{\max}$ a formal combination of an absorbing and reflecting condition, $J + \alpha P_c = 0$, where $\alpha \geq 0$. Suppose $P_{(\lambda)}(\phi)$ is an eigenfunction satisfying $\hat{L}_c P_{(\lambda)} = \lambda P_{(\lambda)}$ with the boundary conditions

$$(J_{(\lambda)} + v_* P_{(\lambda)})|_{\phi_E} = 0, \quad (J_{(\lambda)} + \alpha P_{(\lambda)})|_{\phi_{\max}} = 0, \quad (128)$$

$$J_{(\lambda)} \equiv \partial_\phi(DP_{(\lambda)}) - v_* P_{(\lambda)}. \quad (129)$$

Note that the operator \hat{L}_c is self-adjoint with respect to the scalar product in the “ ψ ” space,

$$(P_1, P_2) \equiv \int \psi_1(x) \psi_2(x) dx = \int P_1(\phi) P_2(\phi) M(\phi),$$

$$M(\phi) \equiv D(\phi) \exp \left[- \int^\phi \frac{v_*(\phi)}{D(\phi)} d\phi \right]. \quad (130)$$

So let us consider the scalar product of $P_{(\lambda)}$ and $\hat{L}_c P_{(\lambda)}$,

$$I[P_{(\lambda)}] \equiv (P_{(\lambda)}, \hat{L}_c P_{(\lambda)}) = \int_{\phi_E}^{\phi_{\max}} P_{(\lambda)} (\hat{L}_c P_{(\lambda)}) M d\phi. \quad (131)$$

By assumption, $\hat{L}_c P_{(\lambda)} = \lambda P_{(\lambda)}$, thus

$$I[P_{(\lambda)}] = \lambda \int_{\phi_E}^{\phi_{\max}} P_{(\lambda)}^2 M d\phi. \quad (132)$$

Since $M(\phi) > 0$, we will prove that $\lambda < 0$ if we show that $I[P_{(\lambda)}] < 0$.

Writing $\hat{L}_c P_{(\lambda)} = \partial_\phi J_{(\lambda)}$ and integrating Eq. (131) by parts, we find

$$I[P_{(\lambda)}] = MP_{(\lambda)} J_{(\lambda)}|_{\phi_E}^{\phi_{\max}} - \int_{\phi_E}^{\phi_{\max}} d\phi J_{(\lambda)} \partial_\phi(MP_{(\lambda)}). \quad (133)$$

Using the boundary conditions (128), we can estimate the boundary term $MP_{(\lambda)} J_{(\lambda)}$ as follows,

$$(MP_{(\lambda)} J_{(\lambda)})|_{\phi_E}^{\phi_{\max}} = -\alpha MP_{(\lambda)}^2(\phi_{\max}) + v_* MP_{(\lambda)}^2(\phi_E) < 0 \quad (134)$$

because $v_*(\phi) < 0$ and $\alpha \geq 0$. It remains to put a bound on the integral in Eq. (133). It is easy to see that $\partial_\phi(MP_{(\lambda)})$ is proportional to $J_{(\lambda)}$, namely

$$\partial_\phi(MP_{(\lambda)}) = \frac{M(\phi)}{D(\phi)} J_{(\lambda)}(\phi). \quad (135)$$

Since $M(\phi)/D(\phi) > 0$, the integral in Eq. (133) must be negative (note that $J_{(\lambda)}(\phi)$ is not everywhere zero):

$$- \int_{\phi_E}^{\phi_{\max}} d\phi J_{(\lambda)} \partial_\phi(MP_{(\lambda)}) = - \int_{\phi_E}^{\phi_{\max}} d\phi J_{(\lambda)}^2 \frac{M}{D} < 0. \quad (136)$$

Thus we have shown that $I[P_{(\lambda)}] < 0$, which proves that every eigenvalue λ of \hat{L}_c is negative.

D. Exit probability in k -inflation

We have seen that the evolution of the averaged field $\bar{\phi}$ is heavily influenced by random “jumps,” which can even bring the value of $\bar{\phi}$ to the upper boundary ϕ_{\max} where the semiclassical description breaks down. Thus, we may trace the evolution of $\bar{\phi}$ along a single comoving trajectory, starting from an initial value ϕ_0 , and ask for the probability p_{exit} of eventually exiting through the end-of-inflation boundary ϕ_E while staying away from the upper boundary ϕ_{\max} . In this section, we show that this exit probability is approximately equal to 1 if ϕ_0 is sufficiently far away from ϕ_{\max} .

The exit probability can be found as follows (a similar method was used in Ref. [22]). Let us assume that the initial distribution is concentrated at $\phi = \phi_0$, i.e.

$$P_c(\phi, t=0) = \delta(\phi - \phi_0), \quad (137)$$

where $\phi_E < \phi_0 < \phi_{\max}$, and that the distribution $P_c(\phi, t)$ is known at all times. Due to the conservation law (127), the probability of exiting inflation through $\phi = \phi_E$ during a time interval $[t, t+dt]$ is

$$dp_{\text{exit}} = Jdt \equiv (\partial_\phi(DP_c) - v_*P_c)|_{\phi=\phi_E} dt = -v_*(\phi_E)P_c(\phi_E, t)dt \quad (138)$$

(note that $v_*(\phi_E) < 0$), hence the total probability of exiting through $\phi = \phi_E$ at any time is

$$p_{\text{exit}} = \int_0^\infty dp_{\text{exit}} = -v_*(\phi_E) \int_0^\infty P_c(\phi_E, t)dt. \quad (139)$$

To compute p_{exit} directly without knowledge of $P_c(\phi, t)$, we define an auxiliary function

$$Q(\phi) \equiv \int_0^\infty dt P_c(\phi, t), \quad (140)$$

so that

$$p_{\text{exit}} = -v_*(\phi_E)Q(\phi_E). \quad (141)$$

It is easy to see that the function $Q(\phi)$ is a solution of

$$\hat{L}_c Q(\phi) = P_c(\phi, t=\infty) - P_c(\phi, t=0) = -\delta(\phi - \phi_0), \quad (142)$$

since $P_c(\phi, t=\infty) = 0$. (Note that the function $v_*(\phi)Q(\phi)$ satisfies a gauge-invariant equation,

$$\partial_\phi \left[\partial_\phi \left(\frac{D}{v_*} (v_* Q) \right) - v_* Q \right] = -\delta(\phi - \phi_0), \quad (143)$$

which reflects the fact that p_{exit} is a gauge-invariant quantity.) The boundary conditions are the time-integrated conditions (128),

$$\partial_\phi(DQ)|_{\phi_E} = 0, \quad Q(\phi_{\max}) = 0, \quad (144)$$

where we have chosen the purely absorbing boundary condition at $\phi = \phi_{\max}$ because we are now interested in regions that never reach that boundary. Rewriting Eq. (142) as

$$\partial_\phi [\partial_\phi(DQ) - v_*Q] = -\delta(\phi - \phi_0), \quad (145)$$

we may immediately integrate,

$$\partial_\phi(DQ) - v_*Q = C_1 - \theta(\phi - \phi_0), \quad (146)$$

where we note that $C_1 = p_{\text{exit}}$ due to the boundary condition at $\phi = \phi_E$. Then we obtain the general solution

$$Q(\phi) = C_2 \frac{1}{D(\phi)} \exp \left[\int_{\phi_E}^{\phi} \frac{v_*}{D} d\phi' \right] + \frac{1}{D(\phi)} \int_{\phi_E}^{\phi} d\phi' (C_1 - \theta(\phi' - \phi_0)) \exp \left[\int_{\phi'}^{\phi} \frac{v_*}{D} d\phi' \right]. \quad (147)$$

The constants of integration $C_{1,2}$ are determined from the boundary conditions (144), and the final result is

$$p_{\text{exit}} = \frac{-\frac{v_*(\phi_E)}{D(\phi_E)} \int_{\phi_0}^{\phi_{\max}} R(\phi') d\phi'}{R(\phi_E) - \frac{v_*(\phi_E)}{D(\phi_E)} \int_{\phi_E}^{\phi_{\max}} R(\phi') d\phi'}, \quad (148)$$

$$R(\phi) \equiv \exp \left[\int_{\phi}^{\phi_{\max}} \frac{v_*}{D} d\phi' \right]. \quad (149)$$

The above equations are valid for any $v_*(\phi)$ and $D(\phi)$; we shall now specialize to the case of k -inflation. Since $v_* < 0$ and is approximately constant, while $D(\phi) > 0$ and grows with ϕ , the function $R(\phi)$ is approximately equal to 1 within a certain (model-dependent) range of ϕ , namely for $\phi_R < \phi < \phi_{\max}$, where ϕ_R is determined by the condition

$$1 \sim \int_{\phi_R}^{\phi_{\max}} \frac{v_*}{D} d\phi \sim \frac{v_0 \phi_R}{D(\phi_R)}. \quad (150)$$

Since $R(\phi)$ quickly approaches zero for $\phi < \phi_R$, we may neglect $R(\phi_E) \ll 1$ and express the exit probability as

$$p_{\text{exit}} = C_1 = \frac{\int_{\phi_0}^{\phi_{\max}} R(\phi) d\phi}{\int_{\phi_E}^{\phi_{\max}} R(\phi) d\phi}. \quad (151)$$

It follows that

$$p_{\text{exit}} \approx 1, \quad \phi_0 < \phi_R; \quad (152)$$

$$p_{\text{exit}} \approx \frac{\phi_{\max} - \phi_0}{\phi_{\max} - \phi_R}, \quad \phi_0 > \phi_R. \quad (153)$$

A worldline starting in the middle of the range $[\phi_R, \phi_{\max}]$ will exit either at $\phi = \phi_E$ or at the upper boundary $\phi = \phi_{\max}$ with nearly equal probability. Therefore, we may interpret the range $[\phi_R, \phi_{\max}]$ as the “runaway diffusion” regime. A typical comoving worldline will surely

(with probability $p_{\text{exit}} \approx 1$) reach the reheating boundary ϕ_E only if the initial value ϕ_0 is outside of the “runaway diffusion” regime, i.e. $\phi_0 < \phi_R$. The value ϕ_R is highly model-dependent, but generically $\phi_R \ll \phi_{\text{max}}$. Therefore, the existence of the “runaway diffusion” regime limits the independence of k -inflation models on initial conditions.

Note that the “runaway diffusion” regime is absent in models of potential-driven inflation, where we have

$$H(\phi) \approx \kappa\sqrt{V}, \quad v_*(\phi) = -\frac{V'}{3H}, \quad D(\phi) = \frac{H^3}{8\pi^2}, \quad (154)$$

and the meaningful range of ϕ is $\phi < \phi_{\text{max}}$, where ϕ_{max} is the Planck boundary determined by $V(\phi_{\text{max}}) \sim M_{\text{Pl}}^4$. One finds

$$R(\phi) = \exp \left[-\frac{8\pi^2}{3\kappa^4} \left(\frac{1}{V(\phi)} - \frac{1}{V(\phi_{\text{max}})} \right) \right], \quad (155)$$

so $R(\phi)$ is negligibly small almost all the way until the boundary $\phi = \phi_{\text{max}}$. It follows that $\phi_R \sim \phi_{\text{max}}$, so the exit probability in potential-driven inflation is $p_{\text{exit}} \approx 1$ for any ϕ_0 within the allowed range $\phi_0 < \phi_{\text{max}}$.

Acknowledgments

The authors are grateful to Florian Marquardt, Slava Mukhanov, Matthew Parry, Ilya Shapiro, and Alex Vikman for stimulating and fruitful discussions.

Appendix A: ATTRACTOR SOLUTIONS IN POTENTIAL-DRIVEN INFLATION

In this appendix, we derive the attractor solution in models of potential-driven inflation with a power-law potential $V(\phi) \propto \phi^n$. This calculation serves as an illustration of the general method for analyzing attractors that we developed in Sec. III. While the existence of attractor behavior in potential-driven inflation is long known, it has not been stressed that a unique attractor solution can be singled out in the phase plane (see e.g. Ref. [17] where a statement is made to the contrary).

We consider a cosmological model of inflation driven by a minimally coupled scalar field ϕ with a potential $V(\phi)$. The equations of motion are

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (A1)$$

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) = \frac{3M_{\text{Pl}}^2}{8\pi}H^2 \equiv \frac{1}{\kappa^2}H^2. \quad (A2)$$

Considering $\dot{\phi}$ as a function of ϕ , i.e. $\dot{\phi} = u(\phi)$, we can reduce the above equations to

$$\frac{du(\phi)}{d\phi} = -\frac{V'(\phi)}{u} - 3\kappa\sqrt{\frac{u^2}{2} + V(\phi)} \equiv g(u, \phi). \quad (A3)$$

For the purposes of further analysis, we assume that the potential $V(\phi)$ has a monotonically growing behavior at $\phi \rightarrow \infty$, going as $V(\phi) \propto \phi^n$ with $n \geq 2$.

1. Determining the attractor solution

According to the criterion developed in Sec. III A, attractors are determined by their behavior at $\phi \rightarrow \infty$. In the phase plane $(\phi, \dot{\phi})$, an attractor solution $\dot{\phi} = u_*(\phi)$ should have the property that neighbor solutions grow significantly faster than $u_*(\phi)$ at $\phi \rightarrow \infty$. In the case of potential-driven inflation with growing $V(\phi)$, we have (at fixed u)

$$g(u, \phi) \sim \max(V', \sqrt{V}) \rightarrow \infty \text{ as } \phi \rightarrow \infty. \quad (A4)$$

Hence, every solution $u(\phi)$ grows at $\phi \rightarrow \infty$. However, a generic trajectory grows exponentially fast for large ϕ , while we expect the attractor trajectory to grow slower than exponentially (e.g. polynomially). To make the attractor behavior apparent, we change variables as $u(\phi) \equiv F(\phi)\tilde{u}(\phi)$, where $F(\phi)$ is a fixed function that will be determined below. The function $F(\phi)$ should have polynomial growth, $F(\phi) \sim \phi^s$, such that the attractor solution is $\tilde{u}(\phi) \rightarrow \text{const}$ at $\phi \rightarrow \infty$.

After the change of variables, the equation for $\tilde{u}(\phi)$ is

$$\begin{aligned} \frac{d\tilde{u}}{d\phi} &= \frac{g(F\tilde{u}, \phi) - F'\tilde{u}}{F} \\ &= -\frac{V'}{F^2\tilde{u}} - 3\kappa\sqrt{\frac{\tilde{u}^2}{2} + \frac{V}{F^2}} - \frac{F'}{F}\tilde{u} \equiv \tilde{g}(\tilde{u}, \phi). \end{aligned} \quad (A5)$$

A solution $\tilde{u}(\phi) \rightarrow \text{const}$ at $\phi \rightarrow \infty$ will exist if $\tilde{g}(\tilde{u}, \phi)$ has an “asymptotic root” $\tilde{u}_0 < 0$ such that

$$\lim_{\phi \rightarrow \infty} \tilde{g}(\tilde{u}_0, \phi) = 0. \quad (A6)$$

Since $F'/F \sim \phi^{-1}$, the last term in Eq. (A5) is always dominated by the second term at fixed \tilde{u} as $\phi \rightarrow \infty$. Therefore, an asymptotic root will exist if the first two terms cancel each other as $\phi \rightarrow \infty$. This requires that one of the two sets of conditions hold in the asymptotic limit $\phi \rightarrow \infty$:

$$\frac{V'}{F^2} \sim \frac{\sqrt{V}}{F}, \quad \frac{\sqrt{V}}{F} \gg 1, \quad (A7)$$

$$\text{or} \quad \frac{V'}{F^2} \sim 1, \quad \frac{\sqrt{V}}{F} \ll 1. \quad (A8)$$

For a power-law potential, $V(\phi) = \lambda\phi^n$, and $F(\phi) = \phi^m$, it is straightforward to see that only the first set of conditions can be met, which yields

$$m = \frac{n}{2} - 1, \quad F(\phi) = \phi^{\frac{n}{2}-1} \sim \frac{V'}{\sqrt{V}}. \quad (A9)$$

The “asymptotic root” \tilde{u}_0 is

$$\tilde{u}_0 = -\frac{n\sqrt{\lambda}}{3\kappa}. \quad (A10)$$

(The value of \tilde{u}_0 is negative since $\dot{\phi} < 0$ for large ϕ .) With the choice $F(\phi) = \phi^{n/2-1}$, the attractor is determined as

the (unique) solution $\tilde{u}_*(\phi)$ which approaches a constant as $\phi \rightarrow \infty$. Other solutions grow exponentially with ϕ .

In the original variables, the attractor solution is

$$u_*(\phi) \approx F(\phi)\tilde{u}_0 \sim \phi^{\frac{n}{2}-1} \text{ as } \phi \rightarrow \infty. \quad (\text{A11})$$

This is the well-known slow-roll attractor behavior in potential-driven inflation, $\dot{\phi} \sim V'/\sqrt{V}$.

2. Approximate expressions for the attractor

A first approximation to the attractor solution is $\tilde{u}_*(\phi) \approx \tilde{u}_0 = \text{const}$, which corresponds (in the original variables) to the solution of the equation

$$u(\phi) = \frac{d\phi}{dt} = -\frac{V'}{3\kappa\sqrt{V}}. \quad (\text{A12})$$

This is the familiar slow-roll approximation. Higher-order asymptotic approximations may be determined by iterating Eq. (A5), considered as an equation for $\tilde{u}(\phi)$ with a given $d\tilde{u}/d\phi$. After some algebra, the next-order approximation to the attractor solution is found as

$$u_*(\phi) \approx -\frac{V'}{3\kappa\sqrt{V}} \left[1 + \frac{4VV'' - 3V'^2}{36\kappa^2 V^2} \right]. \quad (\text{A13})$$

As expected, this expression reproduces the slow-roll expansion of Ref. [17] up to terms of first order.

-
- [1] C. Armendariz-Picon, T. Damour, and V. F. Mukhanov, Phys. Lett. **B458**, 209 (1999), hep-th/9904075.
 - [2] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000), astro-ph/0004134.
 - [3] C. Armendariz-Picon, V. F. Mukhanov, and P. J. Steinhardt, Phys. Rev. **D63**, 103510 (2001), astro-ph/0006373.
 - [4] E. Silverstein and D. Tong, Phys. Rev. **D70**, 103505 (2004), hep-th/0310221.
 - [5] M. Alishahiha, E. Silverstein, and D. Tong, Phys. Rev. **D70**, 123505 (2004), hep-th/0404084.
 - [6] A. Vilenkin, Phys. Rev. **D27**, 2848 (1983).
 - [7] A. A. Starobinsky (1986), in: Current Topics in Field Theory, Quantum Gravity and Strings, Lecture Notes in Physics 206, eds. H.J. de Vega and N. Sanchez (Springer Verlag), p. 107.
 - [8] A. D. Linde, Phys. Lett. **B175**, 395 (1986).
 - [9] A. S. Goncharov, A. D. Linde, and V. F. Mukhanov, Int. J. Mod. Phys. **A2**, 561 (1987).
 - [10] A. D. Linde, D. A. Linde, and A. Mezhlumian, Phys. Rev. **D49**, 1783 (1994), gr-qc/9306035.
 - [11] M. Aryal and A. Vilenkin, Phys. Lett. **B199**, 351 (1987).
 - [12] S. Winitzki, Phys. Rev. **D65**, 083506 (2002), gr-qc/0111048.
 - [13] S. Winitzki, Phys. Rev. **D71**, 123507 (2005), gr-qc/0504084.
 - [14] J. Garcia-Bellido, A. D. Linde, and D. A. Linde, Phys. Rev. **D50**, 730 (1994), astro-ph/9312039.
 - [15] J. Garcia-Bellido and A. D. Linde, Phys. Rev. **D51**, 429 (1995), hep-th/9408023.
 - [16] J. Martin and M. A. Musso, Phys. Rev. **D71**, 063514 (2005), astro-ph/0410190.
 - [17] A. R. Liddle, P. Parsons, and J. D. Barrow, Phys. Rev. **D50**, 7222 (1994), astro-ph/9408015.
 - [18] J. Garriga and V. F. Mukhanov, Phys. Lett. **B458**, 219 (1999), hep-th/9904176.
 - [19] S. Winitzki and A. Vilenkin, Phys. Rev. **D61**, 084008 (2000), gr-qc/9911029.
 - [20] S. Matarrese, M. A. Musso, and A. Riotto, JCAP **0405**, 008 (2004), hep-th/0311059.
 - [21] M. Liguori, S. Matarrese, M. Musso, and A. Riotto, JCAP **0408**, 011 (2004), astro-ph/0405544.
 - [22] A. Vilenkin, Phys. Rev. **D59**, 123506 (1999), gr-qc/9902007.
 - [23] S. Winitzki and A. Vilenkin, Phys. Rev. **D53**, 4298 (1996), gr-qc/9510054.
 - [24] A. D. Linde, D. A. Linde, and A. Mezhlumian, Phys. Lett. **B345**, 203 (1995), hep-th/9411111.
 - [25] A. D. Linde and A. Mezhlumian, Phys. Rev. **D53**, 4267 (1996), gr-qc/9511058.
 - [26] A. Vilenkin, Phys. Rev. Lett. **81**, 5501 (1998), hep-th/9806185.